UNIVERSITY OF VICTORIA
FINAL EXAMINATIONS APRIL 2022

MATH 122: Logic and Foundations

CRN: 22028 (A01), 22029 (A02), 22030 (A03), 22031 (A04)

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Duration: 3 Hours.

Answers are to be written on the exam paper.
No calculator is necessary, but a Sharp EL-510R (plus some letters) calculator is allowed.

This exam consists of 29 questions for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

There are 13 numbered pages, not including covers. Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

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1. [3] Let p, q, r be statements. Suppose that $p \vee q$ is true. Find all truth value assignments of p, q, and r such that $q \to \neg(p \leftrightarrow r)$ is false.

Given $p \vee q$ is T $\therefore$ p is T or q is T or both.

Want $q \to \neg(p \leftrightarrow r)$ is F.

$\therefore$ q is T and $\neg(p \leftrightarrow r)$ F

Since q is T, p can be T or F (b/c $p \vee q$ T)

Need $\neg(p \leftrightarrow r)$ F $\therefore$ $p \leftrightarrow r$ T

$\therefore$ p & r both T or both F

$\therefore$ Solution $\begin{pmatrix} p & q & r \\ T & T & T \\ F & T & F \end{pmatrix}$

2. [4] Use known logical equivalences and the Laws of Logic to prove that $(p \to r) \vee q$ is logically equivalent to $p \to (\neg q \to r)$.

$$\begin{aligned}
(p \to r) \vee q &\Longleftrightarrow (\neg p \vee r) \vee q && \text{Known LE.}\\
&\Longleftrightarrow \neg p \vee (r \vee q) && \text{Assoc}\\
&\Longleftrightarrow \neg p \vee (q \vee r) && \text{Comm}\\
&\Longleftrightarrow p \to (q \vee r) && \text{Known LE}\\
&\Longleftrightarrow p \to (\neg q \to r) && \text{" \quad "}
\end{aligned}$$

3. [2] Indicate on the blank whether each of the following statements are **True (T)** or **False (F)**; no justification is necessary.

T The converse of "If Vancouver wins the Stanley Cup, then Gary owes me \$5" is "If Gary owes me \$5, then Vancouver wins the Stanley Cup".

T The contrapositive of "I eat popcorn only if I am at the movie theatre" is "I don't eat popcorn if I'm not at the movie theatre".

F The negation of "All dogs play frisbee" is "No dogs play frisbee".

F For statements p and q, $p \vee q$ logically implies $p \wedge q$.

4. [4] Use known logical equivalences and inference rules to show that the following argument is valid.

$$\begin{array}{ll} & p \wedge q \\ & \neg(q \wedge \neg r) \\ & (p \wedge r) \to s \\ \hline \therefore & s \end{array}$$

1.	$p \wedge q$	Prem.
2.	$\neg(q \wedge \neg r)$	"
3.	$(p \wedge r) \to s$	"
4.	p	1, Conj. Simpl.
5.	$\neg q \vee \cancel{\neg\neg} r$	2, DeMorgan Dbl Neg'n.
6.	$q \to r$	5, Known LE.
7.	q	1, Conj Simpl.
8.	$\therefore r$	6,7 M.P.
9.	$p \wedge r$	4,8 Conjunction
10	$\therefore s$	3,9 M.P.

5. [2] Give a counterexample to show that the following argument is invalid. Be sure to explain why you have shown that it is invalid.

$$\begin{array}{lll} & p \vee q & ✓ \\ & q \leftrightarrow r & ✓ \\ & r \to s & ✓ \\ \hline \therefore & p \wedge s & ✗ \end{array}$$

$$\left(\begin{array}{cccc} p & q & r & s \\ T & F & F & F \end{array}\right)$$

The given T.A. makes all premises T & the conclusion F. ∴ The argument is not valid.

6. [2] Let $A = \{\emptyset, 1, \{2, 3\}\}$. Indicate on the blank whether each of the following statements are **True (T)** or **False (F)**; no justification is necessary.

F $2 \in A$.

T $\emptyset \in A$.

T $\{1\} \in \mathcal{P}(A)$.

F $|A| = 2$.

7. [2] Consider the universe $\mathcal{U} = \{0, 1, 2, 3\}$. Determine the truth value of the statement $\exists x, \forall y, (y \geq 2) \to ((x + y)^2 \geq 16)$. Explain your reasoning.

True. Take $x=3$.

If $y=0,1$ then $(y \geq 2) \to (x+y)^2 \geq 16$ is T
b/c $y \geq 2$ is F.

If $y=2,3$ then $(y \geq 2) \to (x+y)^2 \geq 16$ is T
b/c $y \geq 2$ is T
and $(x+y)^2 \geq 16$ is T.

∴ When $x=3$ the smt $\forall y, (y \geq 2) \to (x+y)^2 \geq 16$ is T.

8. [4] Let A and B be sets such that $A \cup B \subseteq A$. Prove that $B \subseteq A$, using an argument that starts with "Let $x \in B$..." Then, use the universe $\mathcal{U} = \{1, 2, 3\}$ to give an example that satisfies the statement where the sets A and B are not equal.

Let $x \in B$.
∴ $x \in A \cup B$ by def'n of union.
∴ $x \in A$ b/c $A \cup B \leq A$
∴ $B \leq A$.

Let $A = \{1\}$ and $B = \phi$.
Then $A \cup B \leq A$ and $A \neq B$

9. [2] Let A and B be sets. Indicate on the blank whether each of the following statements are **True (T)** or **False (F)**; no justification is necessary.

T $\{n \in \mathbb{Z} \ : \ 1 \leq n^2 \leq 5\} = \{x \in \mathbb{R} \ : \ (x^2 - 1)(x^2 - 4) = 0\}$. $\{-2,-1,1,2\} \overset{?}{=} \{\pm 1, \pm 2\}$

T $A \oplus B \subseteq A \cup B$.

T If $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$.

F There exists a set with exactly 2 proper subsets.

10. [3] Let A, B, and C be sets. Show that $(A \cup B) \cap (C \cap B^c)^c = (A \setminus C) \cup B$. *Hint:* Use set-theoretic identities (i.e. the Laws of Set Theory).

$(A \setminus C) \cup B = (A \cap C^c) \cup B$ def'n of $A \setminus C$

$= (A \cup B) \cap (B \cup C^c)$ drst.

$= (A \cup B) \cap (B^c \cap C)^c$ DeMorgan

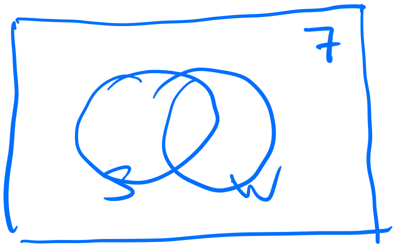
$= (A \cup B) \cap (C \cap B^c)$ comm.

11. [2] In a group of 40 musicians, 22 of them play string instruments, 25 play woodwinds, and 7 of them play neither strings nor woodwinds. How many of the musicians play both strings *and* woodwinds? Be sure to justify your calculations.

Let S be the set of musicians who play strings

& W " " " " " " " " woodwinds

want $|S \cap W|$



$|S \cup W| = |S| + |W| - |S \cap W|$

$\therefore |S \cap W| = |S| + |W| - |S \cup W|$

$= 22 + 25 - 33$

$= 40 - 7$

$= 14$

12. [2] Let $a_0, a_1, a_2, \cdots$ be the sequence given by $a_n = 2n + 5$ for $n \geq 0$. Write a *recursive* definition for this sequence.

$a_0 = 5$

$a_n = a_{n-1} + 2$, $n \geq 1$

13. [3] Let $a_0, a_1, a_2, \cdots$ be the sequence defined recursively by $a_0 = 0$ and $a_n = 2a_{n-1} + 5$ for $n \geq 1$. Express each of a_1, a_2, and a_3 as a sum of terms involving 2s, 5s, and exponents; then, use that work to conjecture a (correct) formula for a_n. You don't need to prove that your formula is correct.

$a_1 = 2a_0 + 5 = 2 \cdot 0 + 5$

$a_2 = 2a_1 + 5 = 2(2 \cdot 0 + 5) + 5 = 2^2 \cdot 0 + 2 \cdot 5 + 5$

$a_3 = 2a_2 + 5 = 2(2^2 \cdot 0 + 2 \cdot 5 + 5) + 5$

$= 2^3 \cdot 0 + 2^2 \cdot 5 + 2 \cdot 5 + 5$

Guess $a_n = \cancel{2^n \cdot 0} + 2^{n-1} \cdot 5 + \cdots + 2 \cdot 5 + 5$

$= 5(1 + 2^1 + \cdots + 2^{n-1}) = \frac{5(2^n - 1)}{2-1} = 5(2^n - 1)$

14. [2] Suppose that we would like to prove that $3^n \geq 2n^3$ for all $n \geq 6$ by induction. After proving that this inequality is true for $n = 6$ directly, we move on to the induction. Write an appropriate Induction Hypothesis to begin the induction portion of the proof. You do not need to continue the proof.

Suppose there exists an integer $k \geq 6$ such that $3^n \geq 2n^3$ for $n = 6, 7, \ldots, k$

15. [2] Indicate on the blank whether each of the following statements are **True (T)** or **False (F)**; no justification is necessary.

T Suppose that $a_n = a_{n-2} - 2a_{n-3}$ for $n \geq 4$. If the terms a_7, a_8, a_9 of the sequence are 6,–4,8, then $a_{10} = -16$.

$a_{10} = a_8 - 2a_7 = -4 - 2 \cdot 6$

F Let $p(n)$ be an open statement in the integer variable n. If you are proving that $p(n)$ is true for all $n \geq 5$ by induction, then you don't need to prove any base cases.

F Define a set X by saying that $2, 5 \in X$ and if $x \in X$, then $2x \in X$. Then $15 \in X$.

T If $S(n)$ is an open statement such that $S(1)$ and $S(2)$ are true and $S(k) \Rightarrow S(k+2)$ for all $k \geq 1$, then $S(n)$ is true for all $n \geq 1$.

16. [5] Let $a_0 = 2$, $a_1 = 1$, and $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$. Use induction to prove that $a_n = 3^n + (-2)^n$ for all $n \geq 0$. Be sure to clearly include all four parts of an induction proof!

Basis. when $n=0$ we have $a_0 = 2 = 3^0 + (-2)^0$ ✓
when $n=1$ we have $a_1 = 1 = 3^1 + (-2)^1$ ✓
$\therefore$ Stmt true when $n=0$ and when $n=1$.

IH. Suppose there exists an integer $k \geqslant 1$ s.t. $a_n = 3^n + (-2)^n$ for $n = 0, 1, \ldots, k$.

IS. Want $a_{k+1} = 3^{k+1} + (-2)^{k+1}$

Look at a_{k+1}. Since $k+1 \geqslant 2$ we have

$$\begin{aligned} a_{k+1} &= a_k + 6a_{k-1} \\ &= (3^k + (-2)^k) + 6(3^{k-1} + (-2)^{k-1}) \quad \text{by IH.} \\ &= 3^k + (-2)^k + 2 \cdot \underbrace{3 \cdot 3^{k-1}} + 2 \cdot 3 \cdot (-2)^{k-1} \\ &= 3^k + (-2)^k + 2 \cdot 3^k - 3 \cdot (-2)^k \\ &= 3 \cdot 3^k - 2 \cdot (-2)^k \\ &= 3^{k+1} + (-2)^{k+1}, \quad \text{as wanted} \end{aligned}$$

$\therefore$ By induction $a_n = 3^n + (-2)^n$ for all $n \geqslant 0$.

17. [4] Prove that for integers a, b, and c, if $a \mid 2b$ and $b \mid 3c$, then $a \mid 12c$.

Suppose $a \mid 2b$ and $b \mid 3c$.

$\therefore$ There exist integers k & ℓ s.t.

$2b = ka$ and $3c = b\ell$

$$\begin{aligned} \therefore \quad 12c &= 4 \cdot 3c \\ &= 4 \cdot b\ell \\ &= 2 \cdot (2b) \cdot \ell \\ &= 2 \cdot (ka) \cdot \ell = (2k\ell)a \end{aligned}$$

Since k, ℓ are integers, so is $2k\ell$.

$\therefore a \mid 12c$

18. [2] Find the base 6 representation of 4783.

$$\begin{aligned} 4783 &= 6 \times 797 + 1 \\ 797 &= 6 \times 132 + 5 \\ 132 &= 6 \times 22 + 0 \\ 22 &= 6 \times 3 + 4 \\ 3 &= 6 \times 0 + 3 \end{aligned}$$

$\therefore 4783 = (34051)_6$

19. [2] Indicate on the blank whether each of the following statements are **True (T)** or **False (F)**; no justification is necessary.

F $(35)_9 = (53)_7$

T The integer $(5A29B678)_{13}$ has a remainder of 8 when divided by 13.

T For a prime p, if $p \mid N$ and $N = (1 \cdot 2 \cdot 3 \cdot \cdots \cdot n) + 1$, then $p > n$.

T The integer $a = 2 \cdot 15^3$ has 9 different positive divisors.

↳ more than 9 $= 2^1 3^3 5^3$ $\therefore$ # div $= 2 \cdot 4 \cdot 4$ $\therefore$ 9

20. [2] Use the Euclidean Algorithm to find $\gcd(3759, 336)$.

$$3759 = 336 \times 11 + 63$$
$$336 = 63 \times 5 + 21 \leftarrow$$
$$63 = 21 \times 3 + 0$$

$\therefore \gcd(3759, 336) = 21$

21. [3] Let k be an integer with the last digit of 9. What is the last digit of $5k^{12} + 13k^3 - 16$? Show your work.

We want r s.t. $0 \le r \le 9$ and

$$5k^{12} + 13k^3 - 16 \equiv r \pmod{10}.$$

Now
$$\begin{aligned} 5k^{12} + 13k^3 - 16 &\equiv 5 \cdot 9^{12} + 13 \cdot 9^3 - 16 \\ &\equiv 5 \cdot (\underset{1}{\cancel{9^2}})^6 + 13 \cdot \underset{1}{\cancel{9^2}} \cdot 9 - 16 \\ &\equiv 5 \cdot 1 + 13 \cdot 9 - 16 \\ &\equiv 5 + 3 \cdot 9 + 4 \\ &\equiv 36 \equiv 6 \pmod{10} \end{aligned}$$

$\therefore$ The last digit is 6.

22. [2] Indicate on the blank whether each of the following statements are **True (T)** or **False (F)**; no justification is necessary.

T For integers a and b, if $a = 2b$ and $\gcd(b, 5) = 1$, then a does not have 5 as a factor in its prime power decomposition.

T If $\gcd(a, b) = 6$, then there exists integers x and y such that $ax + by = 24$.

F If $a^2 \equiv 1 \pmod 5$, then $a \equiv 1 \pmod 5$.

F The last digit (i.e. the ones digit) of 99^{99} is 1.

$$\underline{(99)}^{99} \equiv \underline{9}^{99} \equiv 9 \cdot (\underset{1}{\cancel{9^2}})^{49} \pmod{10}$$

23. Consider the relation $\mathcal{R}$ on the set $\mathbb{Z}$ defined by $(a, b) \in \mathcal{R}$ if and only if $a + 2b$ is an odd integer. Answer each of the following and provide a proof or counterexample as an explanation.

(a) [1] Is $\mathcal{R}$ reflexive?

No. For example $(0,0) \notin R$
b/c $0+2\cdot 0 = 0$,
which is even.

$\therefore$ R not reflexive

(b) [1] Is $\mathcal{R}$ antisymmetric?

No. Since $1+2\cdot 3 = 7$ is odd
$(1,3) \in R$
But $3+2\cdot 1 = 5$, so $(3,1) \in R$
$\therefore$ R not antisymmetric

(c) [2] Is $\mathcal{R}$ transitive?

Yes. Suppose $(a,b),(b,c) \in R$.
Then $a+2b$ is odd and $b+2c$ is odd.
$\therefore$ a is odd
$\therefore$ $a+2c$ is odd (b/c $2c$ is even)
$\therefore$ $(a,c) \in R$ $\therefore$ R is transitive.

24. [2] Indicate on the blank whether each of the following statements are **True (T)** or **False (F)**; no justification is necessary.

T The relation $\mathcal{R}$ on $\mathbb{Z}$ defined by $\mathcal{R} = \{(a, b) : ab \geq 4\}$ is symmetric.
F If a relation $\mathcal{R}$ on a non-empty set A is transitive, then it also is reflexive.
F If $\mathcal{R}$ is an equivalence relation on a non-empty set A, then it is not antisymmetric.
F If $\mathcal{R}$ is a equivalence relation on A, then $\mathcal{R}$ is a function $A \to A$.

25. [3] Let $\mathcal{R}$ be the relation on the set $A = \{10, 11, 12, \ldots, 99999\}$ defined by $a\mathcal{R}b$ exactly when the second-to-last digit of a equals the second-to-last digit of b. (i.e. when the tens digit of a equals the tens digit of b.) It can be shown that $\mathcal{R}$ is an equivalence relation. (You do not need to prove this.) How many equivalence classes does $\mathcal{R}$ have? Explain. Describe the equivalence class that 12201 belongs to.

\# equiv classes

= # possible 2nd last digits

= 10

$[12201]$ = set of #s in A with 10s digit 0

$= \{100, 101, \ldots, 109, 200, 201, \ldots, 209, \ldots, 99900, 99901, \ldots 99909\}$

26. [3] Consider the function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(x) = x^2 - 8$. Determine if $f(x)$ is one-to-one. If it is, provide a proof. If it is not, provide a counterexample.

Suppose $f(x_1) = f(x_2)$

$\therefore \quad x_1^2 - 8 = x_2^2 - 8$

$\therefore \quad x_1^2 = x_2^2$

$\therefore \quad |x_1| = |x_2|$

But $x_1, x_2 \in \mathbb{N}$ so this means $x_1 = x_2$ b/c both are positive

$\therefore$ f is 1-1

27. [3] Let B be the set of odd integers, that is $B = \{2n + 1 : n \in \mathbb{Z}\}$. Consider the function $f : \mathbb{Z} \to B$ defined by $f(x) = 2x + 7$. Determine if $f(x)$ is onto. If it is, provide a proof. If it is not, provide a counterexample.

Let $2n+1 \in B$.

If $f(x) = 2n+1$

then $2x+7 = 2n+1$

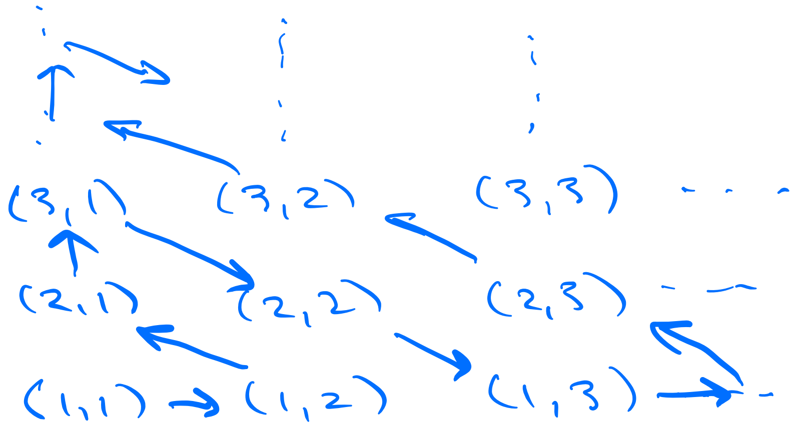
$\therefore \quad 2x = 2n-6$

$\therefore \quad x = (2n-6)/2 = n-3$

Since $n \in \mathbb{Z}$, we have $n-3 \in \mathbb{Z}$

$\therefore$ f is onto

28. [4] Use a diagonal sweeping argument to show that $\mathbb{N} \times \mathbb{N}$ is countable.



Every element of $\mathbb{N} \times \mathbb{N}$ appears in the array.

The seq. indicated by the arrows contains every element of the array & ∴ every element of $\mathbb{N} \times \mathbb{N}$.

∴ $\mathbb{N} \times \mathbb{N}$ is countable

29. [2] Indicate on the blank whether each of the following statements are **True (T)** or **False (F)**; no justification is necessary.

T For non-empty finite sets A and B, if a function $f : A \to B$ is one-to-one but not onto, then $|A| \neq |B|$.

F The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lceil x \rceil$ is onto.

T For a countably infinite set A, if there exists a one-to-one and onto function $f : A \to B$, then B is also countably infinite.

T The set $A = \{n^2 : n \in \mathbb{Z}\}$ is countable.

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