UNIVERSITY OF VICTORIA
EXAMINATIONS APRIL 2023

MATH 122: Logic and Foundations

CRN: 22021 (A01), 22022 (A02), 22023 (A03), 22024 (A04)

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Section and Instructor : ______________________________

NAME: ______________________________

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Duration: 3 Hours.

Answers are to be written on the exam paper.

No calculator is necessary, but a Sharp calculator with model number beginning EL-510R is allowed.

This exam consists of 27 questions, for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [3] Use any method to determine whether $(p \wedge q) \vee (\neg q \to p)$ is a tautology. Write a sentence that explains your conclusion.

Suppose q is F and p is F.

Then $p \wedge q$ is F

& $\neg q \to p$ is F

$\therefore$ $(p \wedge q) \vee (\neg q \to p)$ is F

$\therefore$ The stmt is not a tautology

2. [4] Use known logical equivalences to show that $(p \vee q) \to r$ is logically equivalent to $(p \to r) \wedge (q \to r)$. Give reasons for each step.

$(p \vee q) \to r$

$\Leftrightarrow \neg(p \vee q) \vee r$ Known LE.

$\Leftrightarrow (\neg p \wedge \neg q) \vee r$ DeMorgan

$\Leftrightarrow (\neg p \vee r) \wedge (\neg q \vee r)$ Dist.

$\Leftrightarrow (p \to r) \wedge (q \to r)$ Known LE x2

3. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

T $(p \to q) \wedge (\neg r \to \neg q)$ logically implies $p \to r$.

T The contrapositive of "*I will vote for a candidate only if they wear a big hat*" is "*if a candidate does not wear a big hat, then they will not receive my vote*".

F The negation of "*some of the students laugh when I tell a good joke*" is "*all of the students laugh when I tell a good joke*".

F The converse of "*if an integer is even, then it is not odd*" is "*if an integer is odd, then it is not even.*"

4. [2] Suppose the universe is $\mathcal{U} = \{2, 3, 4, 5\}$. Determine the truth value of the statement $\forall x, \forall y, (x^2 < 3) \to (2y > 7)$. Explain your reasoning.

TRUE. The stmt $x^2 < 3$ is F for every $x \in \mathcal{U}$. $\therefore$ The given implication is T.

5. [4] Use known logical equivalences and inference rules to show that the following argument is valid. Give reasons for each step.

$$\begin{array}{ll} & p \to (q \vee r) \\ & \neg p \to r \\ & \neg r \\ \hline \therefore & q \end{array}$$

1.	$p \to (q \vee r)$	Premise
2.	$\neg p \to r$	"
3.	$\neg r$	"
4.	$\neg r \to p$	2, contrapos
5.	$\therefore p$	3, 4 M.P.
6.	$\therefore q \vee r$	1, 6 M.P.
7	$\neg r \to q$	6, Known LE.
8	$\therefore q$	7, 3, M.P.

6. [3] Give a counterexample to show that the following argument is invalid. Be sure to explain why you have shown that it is invalid.

$$\begin{array}{ll} & q \vee (p \wedge r) \\ & r \to \neg p \\ \hline \therefore & \neg p \end{array}$$

✓ ✗

$\left(\frac{p}{T} \quad \frac{q}{T} \quad \frac{r}{F}\right)$

For the given T.A. all premises are T & the conclusion is F $\therefore$ The argument is not valid.

7. [2] Let $S = \{x, y, \{z\}, \{x, y\}\}$. Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

F $\{y, z\} \subseteq S$.

T $x \in S$.

F $\{y\} \cap \{y, z\} \in S \cap \mathcal{P}(S)$.

T $\emptyset \subseteq S$.

8. Let S and M be sets such that $S \cap M = S \cup M$.

(a) [3] Prove that $S \subseteq M$, using an argument that starts with "*Take any* $x \in S$ ...".

Take any $x \in S$.

$\therefore x \in S \cup M$ by def'n of union.

$\therefore x \in S \cap M$

$\therefore x \in M$ by def'n of intersection

$\therefore S \subseteq M$

(b) [1] Is true that $S = M$? Explain why or why not.

Yes. We show $M \subseteq S$ in the same way as in part (a).

9. [4] Let X and Y be sets. Show that $(X \setminus Y^c) \cup (X^c \cup Y)^c = X$. Hint: there is a short argument that uses set-theoretic identities (remember to give reasons).

$(X \setminus Y^c) \cup (X^c \cup Y)^c$

$= (X \setminus Y^c) \cup (X \cap Y^c)$ DeMorgan

$= (X \cap Y) \cup (X \cap Y^c)$ Known equality

$= X \cap \cancel{(Y \cup Y^c)}\ U$ Dist.

$= X \cap U$ Known.

$= X$ Identity

10. [2] Let A, B and C be sets. Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

F If $A \subseteq B$, then $B \not\subseteq A$.

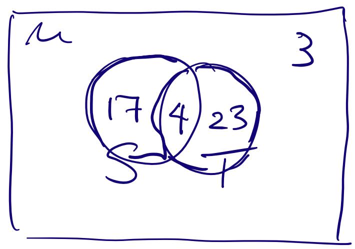
T $A \oplus B = \emptyset$ if and only if $A = B$.

F If $A \cup B = A \cup C$, then $B = C$.

T If A and B are finite sets with $|A| = m$ and $|B| = n$, then $|A \times B| = mn$.

11. [2] In a group of 47 fishers, 21 like catching salmon and 4 like both catching salmon and catching trout. Three fishers like neither catching salmon nor catching trout. How many fishers like exactly one of catching salmon or catching trout?

Let S = set who like catching salmon
T = " " " " trout



From the Venn Diagram $17+23$ fishers like exactly one of catching salmon or catching trout.

12. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 0$, $b_1 = 2$ and $b_n = 4b_{n-1} - 4b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = n2^n$ for all $n \geq 0$.

Basis. When $n=0$ we have $b_0 = 0 = 0 \cdot 2^0$ ✓
& when $n=1$ we have $b_1 = 2 = 1 \cdot 2^1$ ✓
∴ The stmt is true when $n=0$ & when $n=1$.

IH Suppose there is an integer $k \geq 1$ s.t. $b_n = n \cdot 2^n$ for $n = 0, 1, \ldots, k$

IS We want to show $b_{k+1} = (k+1)2^{k+1}$

Consider b_{k+1}. Since $k+1 \geq 2$ we can use the recurrence.

$$\begin{aligned} \therefore b_{k+1} &= 4b_k - 4b_{k-1} \\ &= 4(k \cdot 2^k) - 4((k-1)2^{k-1}) \quad \text{by IH.} \\ &= 2k \cdot 2^{k+1} - (k-1)2^{k+1} \\ &= (2k-(k-1))2^{k+1} \\ &= (k+1)2^{k+1} \quad \text{as we wanted.} \end{aligned}$$

∴ By induction $b_n = n \cdot 2^n \ \forall n \geq 0$

13. [4] Use induction to prove that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$ for all $n\geq 1$.

Basis : when $n=1$, LHS $=\frac{1}{1\cdot 2}$ & RHS $=\frac{1}{2}$

Since LHS = RHS, the stmt is true when $n=1$.

IH Suppose there is an integer $k\geq 1$ s.t.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}$$

IS We want to show

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)\underbrace{\cancel{((k+1)+1)}}_{k+2}}=\frac{(k+1)}{\underbrace{\cancel{((k+1)+1)}}_{k+2}}$$

Consider the LHS

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)}$$

$$=\underbrace{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}}+\frac{1}{(k+1)(k+2)}$$

by meaning of $\cdots$

$$=\frac{(k+2)}{(k+2)}\cdot\frac{k}{(k+1)}+\frac{1}{(k+1)(k+2)} \quad \text{by IH.}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)}=\frac{k+1}{k+2}, \text{ as wanted.}$$

$\therefore$ By induction $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$ $\forall n\geq 1$

14. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

- T If $\lceil x\rceil=\lfloor x\rfloor+1$ then $x\notin\mathbb{Z}$.
- F $(1101010)_2=(C2)_{16}$.
- T For any $n\geq 1$, the integers n and $n+1$ have no common prime factors.
- F If p_1,p_2,p_3,p_4 are different primes, then $\gcd(p_1^1p_2^2p_4^5,\ p_1^3p_3^6p_4^3)=p_1^1p_2^2p_3^6p_4^3$.

15. [3] Let $a_0, a_1, a_2, \ldots$ be the sequence recursively defined by $a_0 = 0$, and $a_n = 5a_{n-1} + 4$ for $n \geq 1$. Express each of a_1, a_2, a_3, a_4 as a sum of terms that involve 4, 5 and exponents. Then, use your work to guess a (correct) formula for a_n. It is not necessary prove that your formula is correct. (Suggestion: first use your work to express a_n as a sum of n terms, as above, and then use that sum to obtain a formula.)

$a_1 = 5a_0 + 4 = 5 \cdot 0 + 4$

$a_2 = 5a_1 + 4 = 5(5 \cdot 0 + 4) + 4 = 5^2 \cdot 0 + 5 \cdot 4 + 4$

$a_3 = 5a_2 + 4 = 5(5^2 \cdot 0 + 5 \cdot 4 + 4) + 4$

$= 5^3 \cdot 0 + 5^2 \cdot 4 + 5 \cdot 4 + 4$

$a_4 = 5a_3 + 4 = 5(5^3 \cdot 0 + 5^2 \cdot 4 + 5 \cdot 4 + 4) + 4$

$= 5^4 \cdot 0 + 5^3 \cdot 4 + 5^2 \cdot 4 + 5 \cdot 4 + 4$

Guess

$a_n = \cancel{5^n \cdot 0} + 5^{n-1} \cdot 4 + 5^{n-2} \cdot 4 + \cdots + 5 \cdot 4 + 4$

$= 4(1 + 5 + 5^2 + \cdots + 5^{n-1})$

$= \dfrac{\cancel{4}(5^{(n-1)+1} - 1)}{\cancel{5-1}} = 5^n - 1$

16. [3] Let $a, b, c \in \mathbb{Z}$. Prove that if $a|b$ and $b|c^2$, then $a^2|bc^2$.

Suppose $a|b$ and $b|c^2$.

$\therefore$ $a|c^2$

$\therefore$ The exist integers k & l s.t.

$ak = b$ and $al = c^2$

$\therefore$ $bc^2 = (ak)(al) = a^2(kl)$

Since $k, l \in \mathbb{Z}$, $kl \in \mathbb{Z}$. $\therefore$ $a^2 | bc^2$

17. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

F $100 = (345)_5$.

T If $a \in \mathbb{Z}$ and $a \equiv 0 \pmod{12}$ then $a \equiv 0 \pmod 6$.

T If a and b are integers such that $7 \mid a$ and $7 \nmid b$, then $a \nmid b$.

T If $a, b \in \mathbb{Z} \setminus \{0\}$, then there exist $x, y \in \mathbb{Z}$ such that $ax + by = 1$ if and only if $\text{lcm}(a, b) = |ab|$.

18. [4] Use the Euclidean Algorithm to find $gcd(1250, 560)$. Then use any method to find $lcm(1250, 560)$.

$$1250 = 560 \cdot 2 + 130$$
$$560 = 130 \cdot 4 + 40$$
$$130 = 40 \cdot 3 + 10 \leftarrow \therefore gcd(1250, 560) = 10$$
$$40 = 10 \cdot 4 + 0$$

$$\therefore lcm(1250, 560) = \frac{1250 \cdot 560}{10}$$
$$= 70{,}000$$

19. [3] Find the last digit of 317^{122}

We want $r \in \{0, 1, \ldots, 9\}$ s.t. $317^{122} \equiv r \pmod{10}$

$$317^{122} \equiv 7^{122} \equiv (7^2)^{61} \equiv 49^{61} \equiv 9^{61} \equiv (-1)^{61}$$
$$\equiv (-1) \equiv 9 \pmod{10}$$

$\therefore$ The last digit is 9.

20. [3] Let n be an integer. Use congruences to show that $12 \mid 17^n - 5^n$.

We want to show $17^n - 5^n \equiv 0 \pmod{12}$

Now $17 \equiv 5 \pmod{12}$

$\therefore 17^n \equiv 5^n \pmod{12}$

$\therefore 17^n - 5^n \equiv 0 \pmod{12}$

$\therefore 12 \mid 17^n - 5^n$

21. [2] Let $X = \{1, 2, 3, 4, 5, 6\}$ and define the relation $\mathcal{R}$ on X by

$$\mathcal{R} = \{(1,1), (1,4), (2,2), (2,3), (2,6), (3,2), (3,3), (3,6), (4,1), (4,4), (5,5), (6,2), (6,3), (6,6)\}.$$

Take it as given that $\mathcal{R}$ is an equivalence relation (do not prove this). What partition of X is determined by the equivalences classes of $\mathcal{R}$? No justification is needed.

$\{ \{1,4\}, \{2,3,6\}, \{5\} \}$

22. [3] Prove that if $\mathcal{R}$ and $\mathcal{S}$ are symmetric relations on the set X, then $\mathcal{R} \cap \mathcal{S}$ is also a symmetric relation on X.

Suppose $(x,y) \in R \cap S$

By def'n of intersection $(x,y) \in R$ and $(x,y) \in S$

Since R is symmetric $(y,x) \in R$

" S " " $(y,x) \in S$

$\therefore (y,x) \in R \cap S$

$\therefore R \cap S$ is symmetric.

23. [2] Use the blank to indicate whether each statement is **True** (**T**) or **False** (**F**). No reasons are necessary.

F For a non-empty set A, the only relation on A which is symmetric and antisymmetric is $\emptyset$.

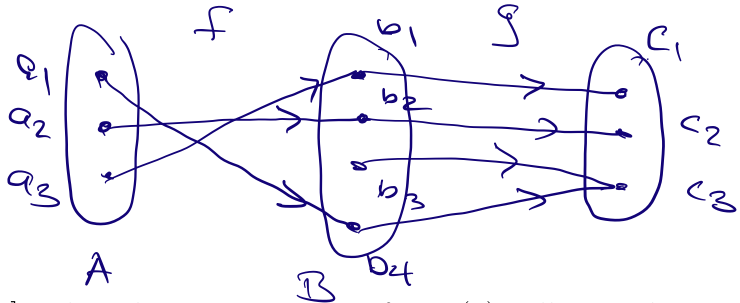
F If $\mathcal{R}$ is a symmetric and transitive relation on a non-empty set X, then $\mathcal{R}$ is reflexive.

F If f is a function $A \to B$ and $|A| < |B|$, then $f : A \to B$ must be onto.

T If $f : A \to B$ is invertible, then $f^{-1} \circ f = \iota_A$, where ι_A is the identity function on A.

24. Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3, b_4\}$, and $C = \{c_1, c_2, c_3\}$.
Define $f : A \to B$ by $f = \{(a_1, b_4), (a_2, b_2), (a_3, b_1)\}$.

(a) [3] Describe a function $g : B \to C$ so that $g \circ f$ is 1-1 and onto. Describe each of g and $g \circ f$ as a set of ordered pairs.



$g = \{(b_1, c_1), (b_2, c_2), (b_3, c_3), (b_4, c_3)\}$

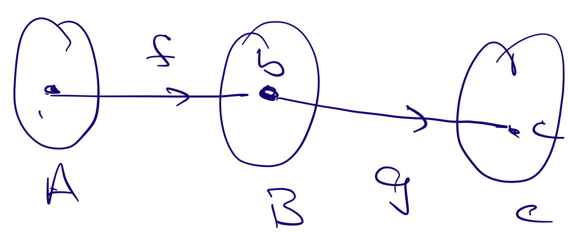
$g \circ f = \{(a_1, c_3), (a_2, c_2), (a_3, c_1)\}$

(b) [1] What does your answer from (a) tell you about the converse of the (true) statement "If $f : A \to B$ and $g : B \to C$ are both 1-1 and onto, then $g \circ f$ is 1-1 and onto". Explain in at most 2 sentences.

It tells me that the converse is false.
In (a), $g \circ f$ is 1-1 & onto but f isn't onto and g isn't 1-1.

25. [3] For sets A, B, and C, let $f : A \to B$ and $g : B \to C$ be onto functions. Prove that $g \circ f : A \to C$ is onto.

Take any $c \in C$. We want to find $a \in A$ s.t. $g \circ f(a) = c$.
Since g is onto, there exists $b \in B$ s.t. $g(b) = c$.
Since f is onto, there exists $a \in A$ s.t. $f(a) = b$.



Now $g \circ f(a) = g(f(a))$
$= g(b) = c$

$\therefore$ $g \circ f$ is onto

26. Let S be the set of all infinite binary sequences (infinite sequences of 0s and 1s). Fill in the proof that S is uncountable as requested in (a), (b), and (c) below.

Suppose to the contrary that S is countable. Then there is a 1-1 and onto function $f : \mathbb{N} \to S$:

$$\begin{array}{rcl} f(1) & = & s_{11}s_{12}s_{13}\ldots \\ f(2) & = & s_{21}s_{22}s_{23}\ldots \\ f(3) & = & s_{31}s_{32}s_{33}\ldots \\ \vdots & \vdots & \vdots \end{array}$$

where, for every natural number n, $f(n)$ is the infinite binary sequence $s_{n1}s_{n2}s_{n3}\ldots$.

(a) [1] Complete the following definition of an infinite binary sequence $b_1b_2\ldots$ which is not $f(n)$ for any natural number n.

$$\text{For } i = 1, 2, \ldots, n, \text{ define } \quad b_i = \begin{cases} 0 & \text{if} \quad s_{ii} = 1 \\ 1 & \text{if} \quad s_{ii} = 0 \end{cases}$$

(b) [1] Explain in at most two sentences why the sequence defined in (a) is not $f(n)$ for any natural number n.

It can't be f(n) for any n because, by def'n, $b_n \neq s_{nn}$.

(c) [2] Explain why the existence of the sequence defined in (a) gives a contradiction and completes the proof. In particular, what is contradicted?

If S were countable, then $b_1 b_2 \ldots$ must be f(n) for some $n \in \mathbb{N}$ (b/c f is onto.)
By (b), $b_1 b_2 \ldots$ is not f(n) for any $n \in \mathbb{N}$
$\therefore$ f is not onto, contradicting that f is onto.

27. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

F $|\{-1, 0, 1\}| = |\{0, 1\}|$

T $|\mathbb{Z}| = |\mathbb{Q}|$

F There is a one-to-one and onto function $f : \mathbb{N} \to (0, 1)$, where (0, 1) is the open interval of real numbers between 0 and 1.

F There exists a set S such that $|\mathcal{P}(S)| = |S|$.

/END