UNIVERSITY OF VICTORIA
APRIL EXAMINATIONS 2024

MATH 122: Logic and Foundations

CRN: 22013 (A01), 22014 (A02), 22015 (A03), 22016 (A04), 24097 (A05)

Please circle your instructor and section:

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NAME: ______________________________

V00#: ______________________________

Duration: 3 Hours.

Answers are to be written on the exam paper.
No calculator is necessary, but a Sharp calculator with model identifier beginning EL-510R is allowed.

This exam consists of 25 questions, for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [3] Use any method to determine whether $p \to (q \to p)$ is a tautology.

p	q	$q \to p$	$p \to (q \to p)$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

The statement $p \to (q \to p)$ is always T
$\therefore$ It is a tautology

2. [4] Use known logical equivalences to show that $\neg(\neg p \wedge \neg q) \wedge (p \vee \neg r)$ is logically equivalent to $p \wedge (r \to q)$.

For the truth assignment $\begin{pmatrix} p & q & r \\ 1 & 0 & 1 \end{pmatrix}$ the 1st statement is T & the second one is false
$\therefore$ The statements are not l.e. and the question has an error

It is likely that what was intended for the 1st statement is $\neg(\neg p \vee \neg q) \vee (p \wedge \neg r)$

$\Leftrightarrow (p \wedge q) \vee (p \wedge \neg r)$ DeMorgan

$\Leftrightarrow p \wedge (q \vee \neg r)$ dist

$\Leftrightarrow p \wedge (r \to q)$ known l.e.

3. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

T If $\neg p \Leftrightarrow \neg q$, then $p \Rightarrow q$.

T For the universe $\mathbb{Z}$, the negation of $\forall x, \exists y, (x \neq 0) \to (xy = 1)$ is $\exists x, \forall y, (x \neq 0) \wedge (xy \neq 1)$.

F The contrapositive of "*If this question is confusing or technical, then it is hard.*" is "*If this question is not hard, then it is not confusing or not technical.*".

F The conclusion of a valid argument is a tautology.

4. [4] Use known logical equivalences and inference rules to show that the following argument is valid.

$$\begin{array}{l} \neg q \vee p \\ \neg r \to q \\ \neg r \\ \hline \therefore \; p \end{array}$$

1.	$\neg q \vee p$	Premise
2.	$\neg r \to q$	"
3.	$\neg r$	"
4.	$\therefore q$	2, 3 M.P.
5.	$q \to p$	1, L.E.
6.	$\therefore p$	5, 4 M.P.

5. [3] Give a counterexample to show that the following argument is invalid.

$$\begin{array}{ll} p \to \neg q & \checkmark \\ q \to \neg s & \\ \neg r \to s & \checkmark \\ p \leftrightarrow r & \checkmark \\ \hline \therefore \; \neg p & F \end{array}$$

$$\begin{pmatrix} p & q & r & s \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$\uparrow$ 1 ok too

The given T.A. makes all premises true & the conclusion false $\therefore$ The argument is not valid.

6. [2] Let A, B and C be sets. Use the blank to indicate whether each statement is True or False. No justification is necessary.

T If $A \subsetneq B$, then $B \neq \emptyset$.

F If $A \subseteq \mathbb{Z}$ and $B \subseteq \mathbb{Q}$, then $A \cap B = \emptyset$.

T If $A \neq \emptyset$ and $B \neq \emptyset$, then $A \times B \neq \emptyset$.

T If $B, C \in \mathcal{P}(A)$ then $B \cup C \in \mathcal{P}(A)$.

7. (a) [3] Let A and B be sets such that $A \cup B \subseteq A \cap B$. Show that $A \subseteq B$ using an argument that starts with "Take any $x \in A$….

Take any $x \in A$.

$\therefore$ $x \in A \cup B$, by def'n of union

$\therefore$ $x \in A \cap B$ since $A \cup B \subseteq A \cap B$

$\therefore$ $x \in B$, by def'n of intersection

$\therefore$ $A \subseteq B$

(b) [1] Must the sets A and B in part (a) be equal? Explain why or why not.

Yes. By the same argument $x \in B \Rightarrow x \in A$

$\therefore$ $x \in A \Leftrightarrow x \in B$.

8. [4] Let A and B be sets. Show that $(A \setminus B^c) \cup (A^c \cup B)^c = A$. Hint: set-theoretic identities.

$(A \setminus B^c) \cup (A^c \cup B)^c$

$= (A \setminus B^c) \cup (A^{\cancel{cc}} \cap B^c)$ DeMorgan & dbl comp.

$= (A \cap B^{\cancel{cc}}) \cup (A \cap B^c)$ Known equality & dbl comp

$= (A \cap B) \cup (A \cap B^c)$ From above

$= A \cap (B \cup B^c)$ Dist.

$= A$ Identity & Known equality

9. [2] Let $A = \{1, 2, 3, 4, 5\}$. Fill in each blank. No justification is necessary.

(a) The number of subsets of A that contain 4 and do not contain 5 is 2^3.

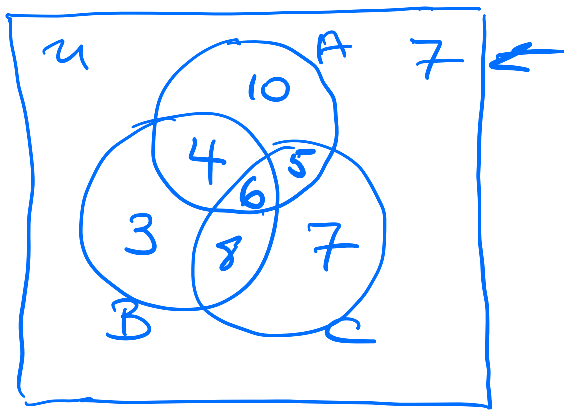
(b) The number of onto functions $f : A \to \{v, w, x, y, z\}$ where $f(1) = x$, $f(2) = w$, $f(3) = z$ and $f(5) \neq v$, is 1.

(c) $|A \times A| =$ 5×5.

(d) The number of relations on A is 2^{25}.

10. [2] On a shelf of 50 books, 25 are non-fiction, 21 have a blue spine, and 26 are written by women. Of these, 10 are non-fiction and have a blue spine, 14 have a blue spine and are written by a woman, 11 are non-fiction books by women, and 6 are non-fiction with a blue spine written by women. How many books on the shelf fit into none of these three categories?

Let A be the set of non-fiction books
B " " " " books with a blue spine
C " " " " " written by women.



We want $50 - |A \cup B \cup C|$
$= 50 - 43$ from diagram
$= 7$

11. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. All variables are integers.

F If $a \mid b$ and $b \mid c$, then $ab \mid c$. $a = b = c = 3$

T If $a = 2^5 \cdot 3^2 \cdot 13^4$ and $b = 2^8 \cdot 11^3$, then $\operatorname{lcm}(a, b) = 2^8 \cdot 3^2 \cdot 11^3 \cdot 13^4$.

T If $\gcd(a, b) = 3$ then there exist integers x and y such that $ax + by = 12$.

T If p is a prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.

12. [4] Use the Euclidean Algorithm to find $d = \gcd(1442, 630)$ and then use your work to find integers x and y such that $1442x + 630y = d$.

$$1442 = 2 \times 630 + 182$$
$$630 = 3 \times 182 + 84$$
$$182 = 2 \times 84 + 14 \leftarrow \text{gcd}$$
$$84 = 6 \times 14 + 0$$

$$\begin{aligned} 14 &= 182 - 2 \times 84 \\ &= 182 - 2 \times (630 - 3 \times 182) \\ &= 7 \times 182 - 2 \times 630 \\ &= 7 \times (1442 - 2 \times 630) - 2 \times 630 \\ &= 7 \times 1442 - 16 \times 630 \end{aligned}$$

$$\therefore \quad 14 = 1442 \times \underbrace{7}_{x} + 630 \underbrace{(-16)}_{y}$$

13. [4] Find the base 7 representation of 2024.

$$2024 = 289 \times 7 + 1$$
$$289 = 41 \times 7 + 2$$
$$41 = 5 \times 7 + 6$$
$$5 = \underline{0} \times 7 + 5$$

$$\therefore \ 2024 = (5621)_7$$

14. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. All variables are integers.

T The last digit of 7^{122} is 9.

F If a is odd, then $a \equiv 1 \pmod 3$.

F If $3a \equiv 3b \pmod{15}$, then $a \equiv b \pmod{15}$.

T $(101011)_2 = (2B)_{16}$.

15. [4] Let $n = (d_2d_1d_0)_{10}$. Prove that $3 \mid n$ if and only if $3 \mid (d_2 + d_1 + d_0)$.

$$\begin{aligned}
3 \mid n &\iff n \equiv 0 \pmod 3 \\
&\iff (d_2d_1d_0)_{10} \equiv 0 \pmod 3 \\
&\iff d_2 10^2 + d_1 \cdot 10 + d_0 \equiv 0 \pmod 3 \\
&\iff d_2 \cdot 1^2 + d_1 \cdot 1 + d_0 \equiv 0 \pmod 3 \\
&\iff d_2 + d_1 + d_0 \equiv 0 \pmod 3 \\
&\iff 3 \mid d_2 + d_1 + d_0
\end{aligned}$$

16. [4] Use induction to prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for all integers $n \geq 1$.

Basis: When $n=1$, LHS $= 1^3 = 1$
& RHS $= 1^2(1+1)^2/4 = 1$
$\therefore$ Stmt true when $n=1$.

IH: Suppose there is an integer $k \geq 1$ s.t. $1^3 + 2^3 + \cdots + n^3 = n^2(n+1)^2/4$ for $n = 1, 2, \ldots, k$

IS: Look at $1^3 + 2^3 + \cdots + (k+1)^3$

Want: this equals $(k+1)^2((k+1)+1)^2/4$
$= (k+1)^2(k+2)^2/4$

$$\begin{aligned}
&1^3 + 2^3 + \cdots + (k+1)^3 \\
&= \underbrace{1^3 + 2^3 + \cdots + k^3} + (k+1)^3 && \text{by meaning of "..."} \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \frac{4}{4} && \text{by IH} \\
&= \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right] \\
&= \frac{(k+1)^2}{4}\left[k^2 + 4k + 4\right] = (k+1)^2(k+2)^2/4
\end{aligned}$$

as wanted

$\therefore$ By induction $1^3 + 2^3 + \cdots + n^3 = n^2(n+1)^2/4$ $\forall n \geq 1$

17. [4] Let $z_1, z_2, \ldots$ be the sequence defined by $z_1 = 1$, and $z_n = z_{n-1} + 2n$ for $n \geq 2$. Find z_2, z_3, z_4 and z_5, then use your work to obtain a formula for z_n (note: a formula, not just a summation). It is not necessary prove that your formula is correct.

$z_1 = 1$

$z_2 = z_1 + 2\cdot 2 = 1 + 2\cdot 2$

$z_3 = z_2 + 2\cdot 3 = 1 + 2\cdot 2 + 2\cdot 3$

$z_4 = z_3 + 2\cdot 4 = 1 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4$

$z_5 = z_4 + 2\cdot 5 = 1 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4 + 2\cdot 5$

Guess: $z_n = 1 + 2\cdot 2 + 2\cdot 3 + \cdots + 2\cdot n$

$= -1 + 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + \cdots + 2\cdot n$

$= -1 + 2(1 + 2 + \cdots + n)$

$= -1 + \dfrac{\cancel{2}\, n(n+1)}{\cancel{2}} = n(n+1) - 1$

18. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 2,\ b_1 = 9$ and $b_n = 9b_{n-1} - 20b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = 4^n + 5^n$ for all $n \geq 0$.

Basis When $n=0$ we have $b_0 = 2 = 4^0 + 5^0$ ✓

when $n=1$ we have $b_1 = 9 = 4^1 + 5^1$ ✓

$\therefore$ Stmt true when $n=0$ & when $n=1$.

IH. Suppose there is an integer $k \geq 1$ s.t. $b_n = 4^n + 5^n$ for $n = 0, 1, \ldots, k$

IS Want: $b_{k+1} = 4^{k+1} + 5^{k+1}$

Look at b_{k+1}. Since $k+1 \geq 1+1 = 2$, we can use the recursion

$\therefore b_{k+1} = 9b_k - 20b_{k-1}$

$= 9(4^k + 5^k) - 20(4^{k-1} + 5^{k-1})$

$= 9\cdot 4^k - 5\cdot 4\cdot 4^{k-1} + 9\cdot 5^k - 4\cdot 5\cdot 5^{k-1}$

$= 9\cdot 4^k - 5\cdot 4^k + 9\cdot 5^k - 4\cdot 5^k$

$= 4\cdot 4^k + 5\cdot 5^k$

$= 4^{k+1} + 5^{k+1}$, as wanted

$\therefore$ By induction $b_n = 4^n + 5^n \ \forall n \geq 0$

19. [4] Let $f : \mathbb{Z} \to \mathbb{N}$ be defined by $f(x) = x^2 + 2$. Explain why f is neither 1-1 nor onto.

f is not 1-1 because $f(-1) = (-1)^2 + 2 = 3$
$= f(1) = 1^2 + 2 = 3$

f is not onto because $x^2 + 2 \geq 2 \;\forall x$
$\therefore \nexists\, x \in \mathbb{Z}$ s.t. $f(x) = 1$

20. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

F For any $x \in \mathbb{R}$, $\lceil x \rceil - 1 = \lfloor x \rfloor$.

F There exists an onto function from $\{1, 2, 3\}$ to $\mathbb{Z}$.

F If $a = b$ implies $f(a) = f(b)$, then the function $f : A \to B$ is 1-1.

T If A and B are finite, and $f : A \to B$ is 1-1 and onto, then $|A| = |B|$.

21. [4] For sets A, B and C, let $f : A \to B$ and $g : B \to C$ be onto functions. Prove that $g \circ f : A \to C$ is onto.

Take any $c \in C$. Want $a \in A$ s.t. $g \circ f(a) = c$

Since g is onto, $\exists\ b \in B$ s.t. $g(b) = c$

Since f is onto, $\exists\ a \in A$ s.t. $f(a) = b$

$\therefore\ g \circ f(a) = g(f(a)) = g(b) = c$

$\therefore\ g \circ f$ is onto

22. [5] Let $A = \{-10, -9, \ldots, -2, -1, 1, 2, \ldots, 9, 10\}$, and let $\sim$ be the relation on A defined by $a \sim b \iff ab > 0$. Prove that $\sim$ is an equivalence relation on A and find the partition of A it determines (i.e., determined by the different equivalence classes with respect to $\sim$).

Reflexive: Let $x \in A$. Then $x^2 > 0$. $\therefore\ x \sim x$

$\therefore \sim$ is reflexive.

Symmetric: Let $a, b \in A$ & suppose $a \sim b$.

$\therefore\ ab > 0$ $\therefore\ ba > 0$ $\therefore\ b \sim a$

$\therefore \sim$ is symmetric

Transitive: Let $a, b, c \in A$ & suppose $a \sim b$ & $b \sim c$

$\therefore\ ab > 0$ & $bc > 0$

$\therefore\ ab^2c > 0$

Since $b^2 > 0$ we can divide.

$\therefore\ \dfrac{a\cancel{b^2}c}{\cancel{b^2}} > \dfrac{0}{b^2} = 0$

$\therefore\ a \sim c$ & $\sim$ is transitive

$\therefore \sim$ is an equivalence relation

The partition of A determined by $\sim$ is $\{\{-1, -2, \ldots, -10\}, \{1, 2, \ldots, 10\}\}$

23. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

F The relation $\{(1,1),(1,2),(1,3),(1,4)\}$ on $\{1,2,3,4\}$ is reflexive.

F For a transitive relation on $\mathbb{Z}$, if $(1,2) \in \mathcal{R}$ and $(2,3) \notin \mathcal{R}$, then $(1,3) \notin \mathcal{R}$.

T The relation $\mathcal{R}$ on $\{2,4,6,\ldots\}$ defined by $(x,y) \in \mathcal{R}$ if and only if $x - y = 2$ is antisymmetric.

T If $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$ such that $1\,\mathcal{R}\,3$ and $4\,\mathcal{R}\,3$, then $4\,\mathcal{R}\,1$.

24. [3] Let $A = \{1,3,5,7,9,\ldots\}$. Use a diagonal sweeping argument to prove that $A \times A$ is countable.

$\vdots \quad\quad \vdots \quad\quad \vdots$

(7,1) ...

(5,1) (5,3) (5,5) ...

(3,1) (3,3) (3,5) ...

(1,1) → (1,3) (1,5) ...

Every element of A×A appears in the array. The sequence indicated by the arrows contains every element of the array & ∴ of A×A

∴ A×A is countable.

25. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

T If A is an uncountable set and $A \subseteq B$, then B is an uncountable set.

T For any real number $b > 0$, the open interval $(0,b)$ is uncountable.

T The closed interval of real numbers $[-1,1]$ is uncountable..

T The set $\{1, -\frac{1}{2}, 3, -\frac{1}{4}, 5, -\frac{1}{6}, 7, -\frac{1}{8}, \ldots\}$ is countable.

/END