UVic – MATH 122 – Logic and Foundations

Date: Apr. 15, 2025

Final Exam

Time Limit: 3 hours

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Before starting your test, enter your name and student ID (clearly) on this page.

> This exam contains 22 pages (including this one, and scrap pages)
> and 27 questions, worth a total of 80 marks.

You may not open the exam until instructed to do so. In the meantime, please read the following instructions:

- No textbooks or class notes are allowed on this exam. A calculator is not necessary, but a Sharp EL-510R, EL-510RN, or EL-510RNB calculator is allowed.

- Answer each question in the space immediately below that question. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

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If you have work on this page you want marked,
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[3] 1. Use any method to determine whether $(p \vee q) \to (p \to q)$ is a tautology. Write a sentence that explains your conclusion.

p	q	$p \vee q$ ①	$p \to q$ ②	① → ②
F	F	F	T	T
F	T	T	T	T
T	F	T	F	F
T	T	T		

when p is true & q is false, the given expression is false ∴ not a tautology

[3] 2. Use laws of logic and known logical equivalences to show that $(p \wedge \neg q) \to r$ is logically equivalent to $p \to (q \vee r)$.

$$(p \wedge \neg q) \to r \iff \neg(p \wedge \neg q) \vee r \qquad \text{Known L.E.}$$
$$\iff (\neg p \vee q) \vee r \qquad \text{DeMorgan, Dbl Neg'n}$$
$$\iff \neg p \vee (q \vee r) \qquad \text{Associative}$$
$$\iff p \to (q \vee r) \qquad \text{Known L.E.}$$

[2] 3. Let A, B, and C be sets. Check the appropriate circle to indicate whether each statement is true or false. No justification is necessary.

True False

(a) ⊗ ○   If $p \to (q \vee r)$ is false, then q is false.

(b) ○ ⊗   If an implication is true, so is its converse.

(c) ⊗ ○   If p and q are logically equivalent, then $\neg p \Rightarrow \neg q$.

(d) ○ ⊗   The negation of the statement "All squares are rectangles." is the statement "There exists a rectangle that is not a square."

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[2] 4. Suppose the universe is $\mathcal{U} = \{1, 2, 3, 4\}$. Determine the truth value of the statement $\exists x, \forall y, (y \geq 3) \to (x^2 < 7)$. Explain your reasoning.

True. Choose $x = 1$. Then if $y \leq 2$ the statement $x^2 < 7$ is true, and if $y \geq 3$ the statement $x^2 < 7$ is true $\therefore$ The statement is true.

[4] 5. Use known logical equivalences and inference rules to show that the following argument is valid.

$$p \to \neg q$$
$$r \vee p$$
$$\underline{ q }$$
$$\therefore r$$

1. $p \to \neg q$ premise
2. $r \vee p$ premise
3. q premise
4. $q \to \neg p$ Contrapos. of 1
5. $\therefore \neg p$ 3,4 MP
6. $\neg p \to r$ 2, Comm. + Known LE.
7. r 5,6 M.P.

[3] 6. Give a counterexample to show that the following argument is invalid. Be sure to explain why what you have shown demonstrates that it is invalid.

$$p \to q \quad T \quad (p \to F)$$
$$\underline{r \to q \quad T \quad (T \to T)}$$
$$\therefore r \to p \quad F \quad (T \to F)$$

$$\left(\begin{array}{ccc} p & q & r \\ F & T & T \end{array} \right)$$

The given truth values make all premises true & the conclusion false $\therefore$ The arg. is not valid.

[2] 7. Check the appropriate circle to indicate whether each statement is true or false. No justification is necessary. Let $A = \{\alpha, \beta, \{\gamma\}, \{\alpha, \gamma\}\}$.

True False

(a)   ⊗   ◯ $\varnothing \subseteq A$.

(b)   ◯   ⊗ $\gamma \in A$.

(c)   ⊗   ◯ $\{\alpha, \beta\} \in \mathcal{P}(A)$.

(d)   ◯   ⊗ $\{\alpha, \gamma\} \subseteq A$.

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[4] 8. Let A and B be sets such that $A \cup B = B$.

(a) Prove that $A \subseteq B$ using an argument that starts with "Let $x \in A$...".

Let $x \in A$.

By def'n of union, $x \in A \cup B$

Since $A \cup B = B$, $x \in B$

(b) Use the universe $\mathcal{U} = \{x, y\}$ to give an example that shows that the sets A and B need not be equal.

$A = \emptyset$, $B = \{x, y\}$. Then $A \cup B$
$= \emptyset \cup \{x, y\} = \{x, y\}$. But $A \neq B$.

[4] 9. Let A, B, and C be sets. Using any method, prove that $A \smallsetminus (B \cap C) = (A \smallsetminus B) \cup (A \smallsetminus C)$.
Hint: There is a short proof using the laws of set theory.

$A \setminus (B \cap C) = A \cap (B \cap C)^c$ definition

$\qquad = A \cap (B^c \cup C^c)$ DeMorgan

$\qquad = (A \cap B^c) \cup (A \cap C^c)$ Distributive

$\qquad = (A \setminus B) \cup (A \setminus C)$ Def'n $\times 2$

[2] 10. Check the appropriate circle to indicate whether each statement is true or false. No justification is necessary. Let A, B, and C be sets.

True False

(a) ⊗ ○   If $A \subsetneq B$, then $B \not\subseteq A$.

(b) ○ ⊗   If $A \cup B = A \cup C$, then $B = C$.

(c) ○ ⊗   If $B \subseteq A \oplus B$, then $A = \emptyset$.

(d) ⊗ ○   If $A \subseteq B$, then $A \times C \subseteq B \times C$.

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[2] 11. Cetatea is an infraorder of acquatic mammals that includes whales, dolphins, and porpoises. Of the 95 species of cetaceans, 33 are under threat, and 6 of the species under threat can be found in the waters of British Columbia. There are 45 species which are neither under threat, nor can be found in BC. How many cetaceans can be found in British Columbia?
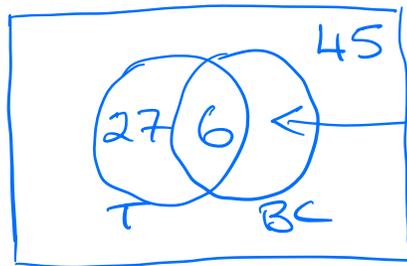
Let T = set under threat
BC = set found in waters of BC.

∴ 17

cetaceans found in BC
$= 17 + 6 = 23$

[3] 12. Let $a, b, c \in \mathbb{Z}$. Prove that if $a \mid b^2$ and $b^4 \mid c^3$, then $a^2 \mid c^3$.

Suppose $a \mid b^2$ ∴ $\exists\, k$ s.t. $ak = b^2$.
Suppose $b^4 \mid c^3$ ∴ $\exists\, \ell$ s.t. $b^4 \ell = c^3$

∴ $c^3 = b^4 \ell = (b^2)^2 \ell = (ak)^2 \ell = a^2 k^2 \ell$
$= a^2 (k^2 \ell)$
Since $k, \ell \in \mathbb{Z}$, $k^2 \ell \in \mathbb{Z}$
∴ $a^2 \mid c^3$

[2] 13. Check the appropriate circle to indicate whether each statement is true or false. No justification is necessary. Let $a, b, c, d \in \mathbb{Z}$.

True False

(a) ⊗ ○ Suppose $a + b = c$. If $d \mid c$ and $d \mid a$, then $d \mid b$. $b = c - a$

(b) ⊗ ○ $\gcd(a, b) \mid \operatorname{lcm}(a, b)$ $\gcd(a,b) \mid a \;\wedge\; a \mid \operatorname{lcm}(a,b)$

(c) ○ ⊗ If $a \mid b^2$, then $a \mid b$.

(d) ⊗ ○ Let p be a prime. If $p \mid ab$, then $p \mid a$ or $p \mid b$.

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[4] 14. Use the Euclidean Algorithm to find $d = \gcd(632, 288)$, and then use your work to find integers x and y such that $632x + 288y = d$.

$$632 = 2 \times 288 + 56$$
$$288 = 5 \times 56 + 8 \leftarrow \gcd(632, 288)$$
$$56 = 7 \times 8 + 0$$

$$8 = 288 - 5 \times 56$$
$$= 288 - 5 \times (632 - 2 \times 288)$$
$$= -5 \times 632 + 11 \times 288$$
$$= 632\underbrace{(-5)}_{x} + 288\underline{(11)}_{y}$$

[2] 15. Find the base 11 representation of 2025. If needed, use α to represent the digit 10.

$$2025 = 184 \times 11 + 1$$
$$184 = 16 \times 11 + 8$$
$$16 = 1 \times 11 + 5$$
$$1 = 0 \times 11 + 1$$

$$\therefore 2025 = (1581)_{11}$$

[2] 16. Check the appropriate circle to indicate whether each statement is true or false. No justification is necessary. Let $a, b \in \mathbb{Z}$.

True False

(a) ⊗ ○   If $\gcd(a, b) = 2$, then there exist $x, y \in \mathbb{Z}$ such that $ax + by = 10$.

(b) ⊗ ○   If $a \equiv 7 \pmod{12}$, then $a \equiv 1 \pmod 3$.

(c) ⊗ ○   If b is even, then $b^2 \equiv 0 \pmod 4$.

(d) ⊗ ○   Let $n \in \mathbb{N}$ and $r \in \mathbb{R}$ with $r > 1$. Then $1 + r + r^2 + \cdots + r^n = \dfrac{r^{n+1} - 1}{r - 1}$.

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[3] 17. Find the last digit of 3^{250} in base 5, that is if $3^{250} = (d_n d_{n-1} \ldots d_0)_5$, find d_0.

Since $3^4 = 81 \equiv 1 \pmod 5$

$3^{250} = 3^2 (3^{248}) = 3^2 ((3^4)^{62})$

$\equiv 3^2 \cdot 1$

$= 9 \pmod 5$

$\equiv 4 \pmod 5$

∴ The last digit is 4.

[4] 18. Use induction to prove that $3^n \geq n^2$ for all $n \geq 2$.

Basis : If $n=2$, $3^n = 3^2 = 9$ and $3^2 = 9$

∴ Statement true when $n=2$.

IH. Suppose there is an integer $k \geq 2$ s.t.

$3^k \geq k^2$ (weak induction)

(or $3^\ell \geq \ell^2$ for $\ell = 2, 3, \ldots, k$)

IS Want $3^{k+1} \geq (k+1)^2$.

Look at $(k+1)^2 = k^2 + 2k + 1$

$\leq k^2 + 2k + k$ b/c $k \geq 2$

$\leq k^2 + k^2 + k^2$ b/c $k \geq 2$

$\leq 3^k + 3^k + 3^k$ by IH

$= 3 \cdot 3^k = 3^{k+1}$, as wanted.

∴ By induction $3^n \geq n^2$ ∀ $n \geq 2$

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[4] 19. Let z_n denote the recursively defined sequence where $z_1 = 0$, and $z_{n+1} = 4z_n + 2$ for all $n \geq 1$.

(a) Find z_2, z_3, z_4, and z_5. You may leave your answers unsimplified.

$$Z_2 = 4Z_1 + 2 = 4 \cdot 0 + 2$$

$$Z_3 = 4Z_2 + 2 = 4 \cdot 2 + 2$$

$$Z_4 = 4Z_3 + 2 = 4(4 \cdot 2 + 2) + 2$$
$$= 4^2 \cdot 2 + 4 \cdot 2 + 2$$

$$Z_5 = 4Z_4 + 2 = 4(4^2 \cdot 2 + 4 \cdot 2 + 2) + 2 = 4^3 \cdot 2 + 4^2 \cdot 2 + 4 \cdot 2 + 2$$

(b) Use your work in (a) to find a formula for z_n that depends only on n, and is valid for all $n \geq 1$. Your answer should not involve a sum of many terms. You do not need to prove that your formula is correct.

Based on (a) guess

$$Z_n = 4^{n-2} \cdot 2 + 4^{n-3} \cdot 2 + \cdots + 4 \cdot 2 + 2$$
$$= 2(1 + 4 + \cdots + 4^{n-2})$$
$$= 2(4^{n-1} - 1)/(4-1) = 2\left(4^{n-1} - 1\right)/3$$

[4] 20. Let a_n denote a sequence recursively defined by $a_1 = 1$, $a_2 = 8$ and $a_n = 6a_{n-1} + 16a_{n-2}$ for all $n \geq 3$. Prove that $a_n = 8^{n-1}$ for all $n \geq 1$.

Basis   When $n=1$, $a_1 = 1$ $\&$ $8^{1-1} = 8^0 = 1$ ✓
         When $n=2$, $a_2 = 8$ $\&$ $8^{2-1} = 8^1 = 8$ ✓
∴ statement is true when $n=1$ $\&$ when $n=2$.

IH. Suppose there exists an integer $k \geq 2$ such that $a_\ell = 8^{\ell-1}$ for $\ell = 1, 2, \ldots, k$

IS   Want to show $a_{k+1} = 8^{k+1-1} = 8^k$.
Look at a_{k+1}. Since $k \geq 2$, $k+1 \geq 3$ $\&$ we can use the recursion
∴ $a_{k+1} = 6a_k + 16a_{k-1} = 6 \cdot 8^{k-1} + 16 \cdot 8^{k-2}$ by IH.
$= 6 \cdot 8 \cdot 8^{k-2} + 16 \cdot 8^{k-2} = 64 \cdot 8^{k-2} = 8^2 \cdot 8^{k-2}$
$= 8^k$ as wanted.   ∴ By induction $a_n = 8^{n-1}$ $\forall n \geq 2$

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[4] 21. Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = x^2 + 1$. Prove that f is $1 - 1$ (injective) but not onto (surjective). Carefully explain your reasoning. Note that $[0, \infty) = \{x : x \geq 0, x \in \mathbb{R}\}$.

$1-1$.   Suppose $f(x_1) = f(x_2)$

∴ $x_1^2 + 1 = x_2^2 + 1$

∴ $x_1 = x_2$ Since f is defined on $[0, \infty)$

∴ f is $1-1$.

not onto: take $y = 0$

Then $f(x) = y$

$\Longleftrightarrow$ $f(x) = 0$ $\Longleftrightarrow$ $x^2 + 1 = 0$

$\Longleftrightarrow$ $x^2 = -1$ which is impossible
in $\mathbb{R}$.

∴ f is not onto.

[2] 22. Check the appropriate circle to indicate whether each statement is true or false. No justification is necessary.

True False

$f = \{(1,1), (2,2), \ldots, (n,n)\}$

(a)   ○   ⊗   A function $f \subseteq A \times A$ is never an equivalence relation.

(b)   ⊗   ○   There are exactly 2^4 possible relations that can be defined on a set of 2 elements. $|A \times A| = 4$ & has 2^4 subsets.

(c)   ○   ⊗   The relation $\mathcal{Q}$ on $\mathbb{Z}$ defined such that $(a, b) \in \mathcal{Q}$ if and only if $a \equiv b \pmod 7$ partitions $\mathbb{Z}$ into 6 equivalence classes.   7

(d)   ○   ⊗   There are no relations on any given set that are simultaneously symmetric and antisymmetric.
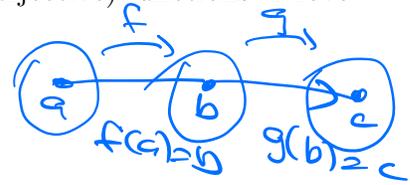
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[4] 23. For sets A, B and C, let $f : A \to B$ and $g : B \to C$ be onto (surjective) functions. Prove that $g \circ f : A \to C$ is onto.

Take any $c \in C$.
Since g is onto, $\exists\ b \in B$
s.t. $g(b) = c$

Since f is onto, $\exists\ a \in A$ s.t. $f(a) = b$.
Then $g \circ f(a) = g(f(a))$
$= g(b)$
$= c$
$\therefore\ g \circ f$ is onto

[4] 24. Define the relation $\mathcal{R}$ on $\mathbb{R}$ such that $(x, y) \in \mathcal{R}$ if and only if $x - y \in \mathbb{Z}$. Prove that $\mathcal{R}$ is an equivalence relation.

Must show reflexive, symmetric & transitive.

reflexive   Take an x. Then $x - x = 0 \in \mathbb{Z}$.
   $\therefore (x, x) \in R$ & R is symmetric.

symmetric   Suppose $x, y \in R$ and $(x, y) \in R$.
   Then $x - y = k \in \mathbb{Z}$.   Then $y - x = -k \in \mathbb{Z}$
   $\therefore (y, x) \in R$ and R is symmetric.

transitive   Suppose $x, y, z \in \mathbb{R}$ and $(x, y), (y, z) \in R$
   Then $x - y = k \in \mathbb{Z}$ and $y - z = \ell \in \mathbb{Z}$
   $\therefore x - z = x - y + y - z = k + \ell \in \mathbb{Z}$. $\therefore (x, z) \in R$
   $\therefore R$ is transitive $\therefore R$ is an equiv. rel'n
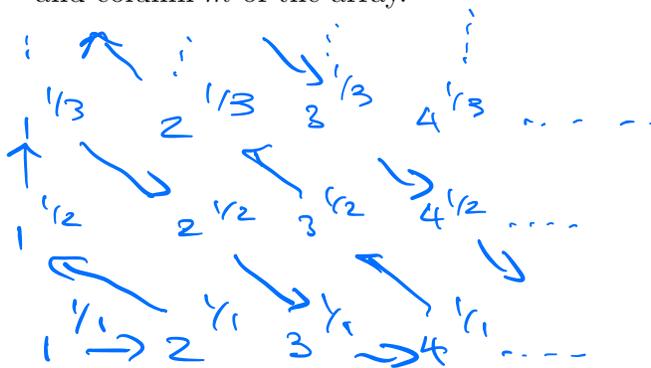
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[2] 25. Consider the set $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$ with the relation $\mathcal{R}$ on A defined by $(a, b) \in \mathcal{R}$ if and only if $a^2 - b^2$ is divisible by 3. The relation $\mathcal{R}$ is an equivalence relation (you do not need to show this). Find all the equivalence classes of $\mathcal{R}$.

0 is related to 0, 3, 6

1 is related to 1, 2, 4, 5, 7

∴ 2 equivalence classes
$\{0, 3, 6\}$, $\{1, 2, 4, 5, 7\}$

[3] 26. Prove that $\{n^{1/m} : n \in \mathbb{N}, m \in \mathbb{N}\}$ is countable using a diagonal sweeping argument.
Hint: In the proof that $\mathbb{N} \times \mathbb{N}$ is countable, the ordered pair (n, m) appeared in row n and column m of the array.

Every number $n^{1/m}$ where $n, m \in \mathbb{N}$ appears in the array (row m & column n)

The sequence indicated by the arrows includes every element of the array i.e. every element of $\{n^{1/m} : n, m \in \mathbb{N}\}$.

∴ The set is countable

[2] 27. Check the appropriate circle to indicate whether each statement is true or false. No justification is necessary.

True False

(a) ○ ⊗(False) Let A be a set with 2 elements and let B be a set with 3 elements. There are exactly two 1-1 (injective) functions $A \to B$.   (3 choices of element to miss)

(b) ⊗(True) ○ A function f has an inverse if and only if the relation $\{(b, a) : (a, b) \in f\}$ is a function.

(c) ○ ⊗(False) The set $\{\sqrt{|x|} : x \in \mathbb{Q}\}$ is uncountable.

(d) ⊗(True) ○ The set of odd integers is countable.

[END OF EXAM]

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