UNIVERSITY OF VICTORIA
EXAMINATIONS DECEMBER 2019

MATH 122: Logic and Foundations

CRN: 12155 (A01), 12156 (A02), 12158 (A04), 12159 (A05)

Instructors: G. MacGillivray (A01), W. Thompson (A02 & A05), K. Mynhardt (A04)

Section and Instructor : ______________________________

NAME: ______________________________

V00#: ______________________________

Duration: 3 Hours.

Answers are to be written on the exam paper.
No calculator is necessary, but a Sharp EL-510R(plus some letters) calculator is allowed.

This exam consists of 26 questions, for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [3] Use any method to determine whether $(\neg p \to q) \wedge (q \leftrightarrow p)$ is a contradiction. Write a sentence that explains your conclusion.

2. [4] Use known logical equivalences to show that $(p \wedge q) \to \neg r$ is logically equivalent to $q \to \neg(p \wedge r)$.

3. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

   ___ q logically implies $p \to q$.

   ___ The negation of "*if you practice regularly then you are good at tennis*" is "*If you don't practice regularly then you are not good at tennis*".

   ___ The converse of "*some Kawasaki motorcycles are fast*" is "*All fast motorcycles are Kawasakis*".

   ___ The contrapositive of "*all mathematicians like logic*" is "*anyone who does not like logic is not a mathematician*".

4. [2] Suppose the universe is $\mathcal{U} = \{1, 2, 3, 4\}$. Determine the truth value of the statement $\exists x, \forall y, (x^2 < 5) \to (y < 2)$. Explain your reasoning.

5. [4] Use known logical equivalences and inference rules to show that the following argument is valid.

$$\begin{array}{ll} & \neg(q \wedge \neg p) \\ & q \vee r \\ & \neg r \\ \hline \therefore & p \end{array}$$

6. [3] Give a counterexample to show that the following argument is invalid. Be sure to explain why you have shown that it is invalid.

$$\begin{array}{ll} & p \to \neg q \\ & q \vee r \\ & \neg r \\ \hline \therefore & p \end{array}$$

7. [2] Let $A = \{a, b, \{c\}, \{a, b\}\}$. Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

___ $c \in A$.

___ $\emptyset \subseteq A$.

___ $\{a, b\} \in A \cap \mathcal{P}(A)$.

___ $|A| = 3$.

8. [4] Let A and B be sets such that $A \setminus B = \emptyset$. Prove that $A \subseteq B$, using an argument that starts with "*Take any* $x \in A$ ...". Then, use the universe $\mathcal{U} = \{1, 2\}$ to give an example that shows the sets A and B need not be equal.

9. [4] Let A, B and C be sets. Show that $(A \cap B) \cup (A \cup B^c)^c = B$. Hint: there is a short argument that uses set-theoretic identities.

10. [2] Let A, B and C be sets. Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

    ___ $A \cap B = A \cup B$ if and only if $A = B$.

    ___ If $A \oplus B = A$, then $B = \emptyset$.

    ___ If $A \cap B = A \cap C$, then $B = C$.

    ___ If $|A| = 3$ and $|B| = 4$, then $|A \times B| = 3^4$.

11. [2] In a group of 32 mathematicians, 25 like coffee and 16 like both coffee and tea. Two mathematicians like neither coffee nor tea. How many mathematicians like tea but not coffee?

12. [3] Let $a, b, c, d \in \mathbb{Z}$. Prove that if $a|b$ and $c|d$, then $ac|bd$.

13. [2] All lower case letters in this question represent integers. Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

___ If p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.

___ $\mathrm{lcm}(2^3 7^2, 2^1 5^2 7^1) = 2^1 5^2 7^1$.

___ $\gcd(a, b) \mid \mathrm{lcm}(a, b)$.

___ If $\gcd(a, b) = 2$, then $\gcd(2a, 2b) = 4$.

14. [4] Use the Euclidean Algorithm to find $d = gcd(1250, 560)$ and then use your work to find integers x and y such that $1250x + 560y = d$.

15. [3] Find the base 6 representation of 2019.

16. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

___ The last digit of 99^{99} is 9.

___ If $a \equiv 0 \pmod{12}$ then $a \equiv 0 \pmod{4}$.

___ If $ak = b$ then every prime divisor of b is a divisor of a.

___ $(1101010)_2 = (6A)_{16}$.

17. Let k be an integer such that $k \equiv -1 \pmod 4$.

    (a) [2] What is the remainder when $7k^3 - 9$ is divided by 4?

    (b) [2] What is the last digit of the base 4 representation of k? Why?

18. [4] Use induction to prove that $0 \cdot 0! + 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ for all integers $n \geq 0$.

19. [3] Let $a_1, a_2, \ldots$ be the sequence recursively defined by $a_1 = 0$, and $a_n = 3a_{n-1} + 7$ for $n \geq 2$. Express each of a_2, a_3, a_4, and a_5 as a sum of terms that involve 3, 7 and exponents. Then, use your work to guess a (correct) formula for a_n. It is not necessary prove that your formula is correct. (Suggestion: first use your work to express a_n as a sum of n terms, as above, and then use that sum to obtain a formula.)

20. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 5,\ b_1 = 10$ and $b_n = 4b_{n-1} - 4b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = 5 \cdot 2^n$ for all $n \geq 0$.

21. [4] Let $f : \mathbb{R} \to [0, \infty)$ be defined by $f(x) = 2x^2$. Show that f is onto but not 1-1. (Note: $[0, \infty) = \{x : x \geq 0\}$.)

22. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

    ___ $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$ for all integers n.

    ___ If $|A| = 122$ and there exists a 1-1 function $f : A \to B$, then $|B| = 122$.

    ___ A function $f : A \to B$ is onto if for every $a \in A$ there exists $b \in B$ such that $(a, b) \in f$.

    ___ For any set A, the identity function on A is invertible.

23. [4] For sets A, B and C, let $f : A \to B$ and $g : B \to C$ be 1-1 functions. Prove that $g \circ f : A \to C$ is 1-1.

24. Let $A = \{1, 2, \ldots, 50\}$. Fill in the blanks. No justification is necessary.

   (a) [1] The number of functions $f : A \to \{1, 2, 3\}$ is ______.

   (b) [1] The number of subsets of $A \times A$ is ______.

   (c) [1] The number of 1-1, onto functions $f : A \to A$ is ______.

   (d) [1] The number of proper subsets of A that contain 50 and do not contain 49 is ______.

25. [3] Use a diagonal sweeping argument to prove that $\mathbb{N} \times \mathbb{N}$ is countable.

26. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

___ Any subset of $\{2, 4, 6, 8, \ldots\}$ is countable.

___ If $a < b$ then the closed interval $[a, b]$ of real numbers is uncountable.

___ The intersection of any two uncountable sets is uncountable.

___ For any set A there exists a set B such that $|B| > |A|$.

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