UNIVERSITY OF VICTORIA
DECEMBER EXAMINATIONS 2014

MATH 122: Logic and Foundations

CRN: 12168 (A01) and 12169 (A02)

INSTRUCTORS: G. MacGillivray (A01) and M. Edwards (A02)

NAME: ______________________________

V00#: ______________________________

Duration: 3 Hours.

Answers are to be written on the exam paper.
No calculator is necessary, but a Sharp EL-510R or a Sharp EL-510RNB calculator is allowed.

This exam consists of 25 questions, for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [3] Find all truth values for p, q and r for which $(p \to \neg q) \leftrightarrow r$ is true.

2. [4] Use known logical equivalences to show that $\neg(\neg p \to \neg q) \vee (p \wedge q)$ is logically equivalent to q.

3. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

   ___ It is possible for a quantified statement and its negation to both be true.

   ___ The negation of $\exists x, \forall y, x \to y$ is $\forall y, \exists x, x \wedge \neg y$.

   ___ The contrapositive of "*Every Honda motorcycle is reliable and handles well.*" is "*Any motorcycle that is unreliable and handles poorly is not a Honda.*".

   ___ The statement $\exists x,\ (x^2 < 0) \to (x = 2)$ is true for the universe $\mathbb{R}$.

4. [4] Use known logical equivalences and inference rules to prove the inference rule called Resolution.

$$\begin{array}{rr} & p \vee r \\ & q \vee \neg r \\ \hline \therefore & p \vee q \end{array}$$

5. [3] Give a counterexample to show that the following argument is invalid.

$$\begin{array}{rr} & p \to \neg q \\ & \neg r \to q \\ \hline \therefore & \neg r \to p \end{array}$$

6. [2] Let A, B, C and D be sets. Use the blank to indicate whether each statement is True or False. No justification is necessary.

___ If $A \times B = B \times A$, then $A = B$, or $A = \emptyset$ or $B = \emptyset$.

___ For any set A, $(\emptyset, \emptyset) \in A \times A$.

___ $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$.

___ If $A \cup B = A \cup C$, then $B = C$.

7. [4] Suppose that $A \subseteq B$. Give an argument that starts with "*Take any* $x \in A \cup B$ ..." to show that $A \cup B \subseteq B$. Are the sets $A \cup B$ and B actually equal? Explain.

8. [4] Is it true that $A \cup (B^c \cap C) = (A \cup B^c) \cap C$ for all sets A, B and C? Give a proof or counterexample, as appropriate.

9. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

   ___ For any set A, there is a relation on A that is reflexive, symmetric, antisymmetric, and transitive.

   ___ For the set $A = \{1, 2, 3, 4\}$, there exists a relation on A that contains $(2, 4)$ and is both symmetric and antisymmetric.

   ___ The number of reflexive relations on $A = \{1, 2, 3, 4, 5\}$ is 2^5.

   ___ If $\mathcal{R}$ and $\mathcal{S}$ are symmetric relations on a set A, then $\mathcal{R} \cup \mathcal{S}$ is a symmetric relation on A.

10. [5] Let $A = \{1, 2, \ldots, 14\}$, and let $\sim$ be the relation on A defined by $a \sim b \Leftrightarrow 4 \mid (a - b)$. Prove that $\sim$ is an equivalence relation on A and find the partition of A it determines.

11. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

    ___ For any $x \in \mathbb{R}$, $\lfloor 2x + 3 \rfloor = 2\lfloor x \rfloor + 3$.

    ___ If a function $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ has the property that $(f \circ f)(x) = x$ for all $x \in \{1, 2, 3, 4\}$, then $f^{-1} = f$.

    ___ The relation $\{(x, y) : y^2 = x + 3\}$ is a function from $\mathbb{R}$ to $(3, \infty)$.

    ___ A function $f : A \to B$ has an inverse if and only if it is a 1-1 correspondence.

12. [4] Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^4 + 4x^2 - 4$. Demonstrate that f is neither 1-1 nor onto.

13. [4] Let $f : A \to B$ and $g : B \to C$ be onto functions. Prove that $g \circ f : A \to C$ is onto.

14. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

    ___ Every subset of an uncountable set is uncountable.

    ___ If A is a countable set and B is an uncountable set, then $A \cup B$ is countable.

    ___ Any non-empty open interval of real numbers is uncountable.

    ___ If A is a countable set then, for any set B, the set $A \cap B$ is countable.

15. [4] Use Cantor Diagonalization to prove that the set $\mathcal{S}$ of all infinite sequences of elements of $\{a, b\}$ is uncountable.

16. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

___ For integers a, b, and c, if $a|bc$, then $a|b$ or $b|c$.

___ For any integer n, the integers n and $n+3$ are relatively prime.

___ $lcm(2^3 11^2, 2^1 7^4) = 2^3 7^4 11^2$.

___ If $a, b \in \mathbb{Z}$,and there exist integers x and y such that $ax + by = 3$, then $gcd(a, b) = 3$.

17. [4] Use the Euclidean Algorithm to find $d = gcd(578, 442)$ and then use your work to find integers x and y such that $578x + 442y = d$.

18. [4] Find the base 5 representation of 1984.

19. [2] Use the blank to indicate whether each statement is True or False. All variables are integers. No justification is necessary.

___ The last digit of 101^{101} is 1.

___ If $ak = b$ then every prime divisor of a is a divisor of b.

___ $(110101)_2 = (35)_{16}$.

___ $(d_2d_1d_0)_{10} \equiv d_2 + d_1 + d_0 (\text{mod } 3)$.

20. [4] Let $a_1, a_2, \ldots$ be the sequence defined by $a_1 = 3$, and $a_n = 2a_{n-1} + 3$ for $n \geq 2$. Find a_2, a_3, a_4 and a_5, then use your work to obtain a formula for a_n. It is not necessary prove that your formula is correct.

21. [4] Use induction to prove that $3^n > n^2$ for all $n \geq 1$.

22. [3] Use the Fundamental Theorem of Arithmetic to explain why there are no integers a and b such that $3b^2 = a^2$, and then use this fact to prove by contradiction that $\sqrt{3}$ is irrational.

23. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 2,\ b_1 = 1$ and $b_n = b_{n-1} + 2b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = 2^n + (-1)^n$ for all $n \geq 0$.

24. [2] When Christi and Gary go out for dinner it is either just the two of them, or the two of them together with one or both of their two closest friends. This term they have gone out with each of these friends a total of 7 times, and with both of them together 3 times. If, over the term, Christi and Gary have gone out for dinner a total of 18 times, how many times have the two of them gone out for dinner together with neither of their closest friends?

25. [2] Let $A = \{1, 2, 3, 4, 5, 6\}$. Fill in each blank. No justification is necessary.

    (a) $|A \times A| =$ ____________________.

    (b) The number of subsets of A that contain none of 1, 2, and 3 is ____________________.

    (c) The number of functions $f : A \to \{w, x, y, z\}$ where $f(1) = w$, $f(2) = z$, $f(5) = x$, and $f(6) \neq y$ is ____________________.

    (d) The number of relations on A is ____________________.

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