1. [3] Find all truth values for p, q and r for which $(p \to \neg q) \leftrightarrow r$ is true.

Need both T or both F

$$\begin{pmatrix} p & q & r \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

r T $\therefore$ need $p \to \neg q$ T
$\therefore$ p F
or p T, $\neg q$ T

r F $\therefore$ need $p \to \neg q$ F
$\therefore$ p T, $\neg q$ F.

2. [4] Use known logical equivalences to show that $\neg(\neg p \to \neg q) \vee (p \wedge q)$ is logically equivalent to q.

$\neg(\neg p \to \neg q) \vee (p \wedge q)$	
$\Leftrightarrow \neg(\cancel{\neg\neg} p \vee \neg q) \vee (p \wedge q)$	Known L.E. Dbl Neg'n.
$\Leftrightarrow (\neg p \wedge \cancel{\neg\neg} q) \vee (p \wedge q)$	DeMorgan's Dbl Neg'n.
$\Leftrightarrow \underbrace{(\neg p \vee p)}_{1} \wedge q$	Dist've Known Tautology
$\Leftrightarrow q$	Identity Law.

3. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

F It is possible for a quantified statement and its negation to both be true.

F The negation of $\exists x, \forall y, x \to y$ is $\forall y, \exists x, x \wedge \neg y$.

F The contrapositive of "*Every Honda motorcycle is reliable and handles well.*" is "*Any motorcycle that is unreliable and handles poorly is not a Honda.*".

T The statement $\exists x,\ (x^2 < 0) \to (x = 2)$ is true for the universe $\mathbb{R}$.

4. [4] Use known logical equivalences and inference rules to prove the inference rule called Resolution.

$$\begin{array}{rr} & p \vee r \\ & q \vee \neg r \\ \hline \therefore & p \vee q \end{array}$$

1. $p \vee r$ premise
2. $q \vee \neg r$ premise
3. $\neg q \to \neg r$ L.E. to 2
4. $r \vee p$ 1, Commutative
5. $\neg r \to p$ L.E. to 4
6. $\neg q \to p$ 3, 5, Chain Rule
7. $\neg\neg q \vee p$ L.E. to 6
8. $p \vee q$ Comm & Dbl Neg'n.

5. [3] Give a counterexample to show that the following argument is invalid.

$\neg r \to p$ F
$\therefore$ r F & p F
$\neg r \to q$ T
$\therefore$ q T

$$\begin{array}{rrl} & p \to \neg q & \text{T} \\ & \neg r \to q & \text{T} \\ \hline \therefore & \neg r \to p & \text{F} \end{array}$$

The truth assignment $\begin{pmatrix} p & q & r \\ 0 & 1 & 0 \end{pmatrix}$ makes all premises true & the conclusion false

6. [2] Let A, B, C and D be sets. Use the blank to indicate whether each statement is True or False. No justification is necessary.

T If $A \times B = B \times A$, then $A = B$, or $A = \emptyset$ or $B = \emptyset$.

F For any set A, $(\emptyset, \emptyset) \in A \times A$.

T $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$.

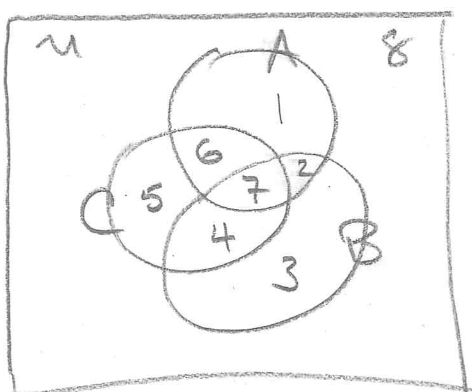
F If $A \cup B = A \cup C$, then $B = C$.

7. [4] Suppose that $A \subseteq B$. Give an argument that starts with "*Take any* $x \in A \cup B$ ..." to show that $A \cup B \subseteq B$. Are the sets $A \cup B$ and B actually equal? Explain.

Take any $x \in A \cup B$.
$\therefore$ $x \in A$ or $x \in B$.
But if $x \in A$, then $x \in B$ b/c $A \subseteq B$.
$\therefore$ In either case $x \in B$
$\therefore$ $A \cup B \subseteq B$.

Yes. It is always true that $B \subseteq A \cup B$
$\therefore$ If $A \subseteq B$, then $A \cup B = B$.

8. [4] Is it true that $A \cup (B^c \cap C) = (A \cup B^c) \cap C$ for all sets A, B and C? Give a proof or counterexample, as appropriate.



$A = \{1,2,6,7\}$
$B = \{2,3,4,7\}$ $B^c = \{1,5,6,8\}$
$C = \{4,5,6,7\}$
$B^c \cap C = \{5,6\}$
LHS $= \{1,2,5,6,7\}$
$A \cup B^c = \{1,2,5,6,7,8\}$
RHS $= \{5,6,7\} \neq$ LHS

$\therefore$ The sets are not equal.
The counterexample is above

9. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

T For any set A, there is a relation on A that is reflexive, symmetric, antisymmetric, and transitive.

F For the set $A = \{1, 2, 3, 4\}$, there exists a relation on A that contains $(2, 4)$ and is both symmetric and antisymmetric.

F The number of reflexive relations on $A = \{1, 2, 3, 4, 5\}$ is 2^5.

T If $\mathcal{R}$ and $\mathcal{S}$ are symmetric relations on a set A, then $\mathcal{R} \cup \mathcal{S}$ is a symmetric relation on A.

10. [5] Let $A = \{1, 2, \ldots, 14\}$, and let $\sim$ be the relation on A defined by $a \sim b \Leftrightarrow 4 \mid (a - b)$. Prove that $\sim$ is an equivalence relation on A and find the partition of A it determines.

Reflexive: Let $x \in A$. Then $x - x = 0$ & $4 \mid 0 = x - x$
$\therefore x \sim x$ $\therefore \sim$ is reflexive.

Symmetric: Suppose $x \sim y$. Then $4 \mid x - y$
$\therefore \exists\, k \in \mathbb{Z}$ s.t. $4k = x - y$
$\therefore 4(-k) = y - x$
$\therefore 4 \mid y - x$ $\therefore y \sim x$
$\therefore \sim$ is symmetric

Transitive: Suppose $x \sim y$ & $y \sim z$
$\therefore 4 \mid x - y$ & $4 \mid y - z$
$\therefore 4 \mid (x - y) + (y - z) = x - z$
$\therefore \sim$ is transitive

$\therefore \sim$ is an equivalence relation.
The partition of A determined by $\sim$ is
$\{\{1, 5, 9, 13\}, \{2, 6, 10, 14\}, \{3, 7, 11\}, \{4, 8, 12\}\}$

11. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

F For any $x \in \mathbb{R}$, $\lfloor 2x + 3 \rfloor = 2\lfloor x \rfloor + 3$.

T If a function $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ has the property that $(f \circ f)(x) = x$ for all $x \in \{1, 2, 3, 4\}$, then $f^{-1} = f$.

F The relation $\{(x, y) : y^2 = x + 3\}$ is a function from $\mathbb{R}$ to $(3, \infty)$.

T A function $f : A \to B$ has an inverse if and only if it is a 1-1 correspondence.

12. [4] Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^4 + 4x^2 - 4$. Demonstrate that f is neither 1-1 nor onto.

Since $f(1) = 1^4 + 4 \cdot 1^2 - 4 = 1$
& $f(-1) = (-1)^4 + 4(-1)^2 - 4 = 1$
But $1 \neq -1$, f is not 1-1.

Consider $y = -5$. Then $f(x) = -1$
$\iff x^4 + 4x^2 - 4 = -5 \iff x^4 + 4x^2 = -1$
$\iff x^4 + 4x^2 + 1 = 0$
The LHS ≥ 1 $\therefore$ There is no x s.t. $x^4 + 4x^2 + 1 = 0$ $\therefore$ f is not onto

13. [4] Let $f : A \to B$ and $g : B \to C$ be onto functions. Prove that $g \circ f : A \to C$ is onto.

Take any $c \in C$.
Since g is onto, $\exists\, b \in B$ s.t. $g(b) = c$.
Since f is onto, $\exists\, a \in A$ s.t. $f(a) = b$.

$\therefore$ $g \circ f(a) = g(f(a))$
$= g(b) = c$

$\therefore$ $g \circ f$ is onto

14. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

F Every subset of an uncountable set is uncountable.
F If A is a countable set and B is an uncountable set, then $A \cup B$ is countable.
T Any non-empty open interval of real numbers is uncountable.
T If A is a countable set then, for any set B, the set $A \cap B$ is countable.

15. [4] Use Cantor Diagonalization to prove that the set $\mathcal{S}$ of all infinite sequences of elements of $\{a, b\}$ is uncountable.

Suppose S is countable. Then there is a list that contains every element of S:

1. $\boxed{s_{11}}$ s_{12}, s_{13}, s_{14}, ...
2. s_{21}, $\boxed{s_{22}}$ s_{23}, s_{24}, ...
3. $\vdots$ $\vdots$ $\boxed{}$

Define the seq $x = x_1, x_2, \ldots$ by

$$x_i = \begin{cases} a & \text{if } s_{ii} = b \\ b & \text{if } s_{ii} = a \end{cases}.$$

Then $x \in S$. But x is not in the list. Suppose it is. Then it has a position, say n. Therefore

$$x_1 = s_{n1}, \quad x_2 = s_{n2}, \quad \ldots \text{ i.e., } x_i = s_{ni} \ \forall i$$

But, by def'n $x_n \neq s_{nn}$ $\Rightarrow\Leftarrow$

$\therefore$ S is not countable

16. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

F For integers a, b, and c, if $a|bc$, then $a|b$ or $b|c$.

F For any integer n, the integers n and $n+3$ are relatively prime.

T $lcm(2^3 11^2, 2^1 7^4) = 2^3 7^4 11^2$.

F If $a, b \in \mathbb{Z}$,and there exist integers x and y such that $ax + by = 3$, then $gcd(a, b) = 3$.

17. [4] Use the Euclidean Algorithm to find $d = gcd(578, 442)$ and then use your work to find integers x and y such that $578x + 442y = d$.

$578 = 442 \times 1 + 136$
$442 = 136 \times 3 + \boxed{34} \leftarrow gcd(578, 442)$
$136 = 34 \times 4 + 0$

$34 = 442 - 136 \times 3$
$= 442 - (578 - 442) \times 3$
$= 578\underbrace{(-3)}_{x} + 442\underbrace{(4)}_{y}$

18. [4] Find the base 5 representation of 1984.

$1984 = 5 \times 396 + 4$
$396 = 5 \times 79 + 1$
$79 = 5 \times 15 + 4$
$15 = 5 \times 3 + 0$
$3 = 5 \times 0 + 3$

$\therefore 1984 = (30414)_5$

19. [2] Use the blank to indicate whether each statement is True or False. All variables are integers. No justification is necessary.

T The last digit of 101^{101} is 1.
T If $ak = b$ then every prime divisor of a is a divisor of b.
T $(110101)_2 = (35)_{16}$.
T $(d_2d_1d_0)_{10} \equiv d_2 + d_1 + d_0 (\text{mod } 3)$.

20. [4] Let $a_1, a_2, \ldots$ be the sequence defined by $a_1 = 3$, and $a_n = 2a_{n-1} + 3$ for $n \geq 2$. Find a_2, a_3, a_4 and a_5, then use your work to obtain a formula for a_n. It is not necessary prove that your formula is correct.

$a_1 = 3$

$a_2 = 2a_1 + 3 = 2 \cdot 3 + 3 = 9$

$a_3 = 2a_2 + 3 = 2(2 \cdot 3 + 3) + 3 = 2^2 \cdot 3 + 2 \cdot 3 + 3 = 21$

$a_4 = 2a_3 + 3 = 2(2^2 \cdot 3 + 2 \cdot 3 + 3) + 3$
$= 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3 = 45$

$a_5 = 2a_4 + 3 = 2(2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3) + 3$
$= 2^4 \cdot 3 + 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3 = 93$

Guess $a_n = 2^{n-1} \cdot 3 + 2^{n-2} \cdot 3 + \cdots + 2 \cdot 3 + 3$
$= 3(2^{n-1} + 2^{n-2} + \cdots + 1) = 3\left(\frac{2^n - 1}{2 - 1}\right)$
$= 3(2^n - 1)$

21. [4] Use induction to prove that $3^n > n^2$ for all $n \geq 1$.

Basis: When $n = 1$, LHS $= 3^1 = 3$, RHS $= 1^2 = 1$. $\therefore$ stmt true when $n = 1$

when $n = 2$, LHS $= 3^2 > 2^2 =$ RHS.
When $n = 3$, LHS $= 3^3 > 3^2 =$ RHS.
$\therefore$ Stmt true when $n = 1, n = 2$ or $n = 3$.

IH: Assume $3^k > k^2$ for some $k \geq 3$

IS: Goal: show $3^{k+1} > (k+1)^2$

Consider $(k+1)^2 = k^2 + 2k + 1$
$< k^2 + 2k + k$ $(k \geq 3)$
$= k^2 + 3 \cdot k$
$\leq k^2 + k \cdot k$ $(k \geq 3)$
$< 3^k + 3^k$ by IH
$= 2 \cdot 3^k < 3 \cdot 3^k = 3^{k+1}$ ✓

$\therefore$ By induction, $3^n > n^2$ for all $n \geq 1$

22. [3] Use the Fundamental Theorem of Arithmetic to explain why there are no integers a and b such that $3b^2 = a^2$, and then use this fact to prove by contradiction that $\sqrt{3}$ is irrational.

By FTA, LHS & RHS have the same prime factors. But the exponent of 3 is odd on the LHS & even on the RHS.
$\therefore$ $3b^2$ can't equal a^2

Suppose $\sqrt{3}$ is rational $\therefore \exists\, a, b \in \mathbb{Z}$ s.t. $\sqrt{3} = a/b$. $\therefore 3 = a^2/b^2$ or $3b^2 = a^2$.
By the 1st part, this is a contradiction
$\therefore$ $\sqrt{3}$ is not rational

23. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 2,\ b_1 = 1$ and $b_n = b_{n-1} + 2b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = 2^n + (-1)^n$ for all $n \geq 0$.

Basis When $n=0$, $b_0 = 2 = 2^0 + (-1)^0 = 1+1$
when $n=1$, $b_1 = 1 = 2^1 + (-1)^1 = 2-1$
$\therefore$ Stmt true when $n=0$ & when $n=1$.

IH Assume $b_0 = 2^0 + (-1)^0$, $b_1 = 2^1 + (-1)^1, \ldots,$
$b_k = 2^k + (-1)^k$ for some $k \geq 1$.

IS Goal: $b_{k+1} = 2^{k+1} + (-1)^{k+1}$.

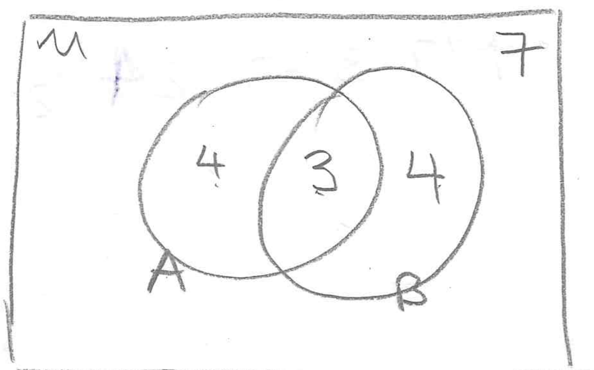
Consider b_{k+1}. Since $k+1 \geq 2$,

$$
\begin{aligned}
b_{k+1} &= b_k + 2b_{k-1} \\
&= 2^k + (-1)^k + 2\cdot(2^{k-1} + (-1)^{k-1}) \quad \text{by IH} \\
&= 2^k + (-1)^k + 2^k + 2(-1)^{k-1} \\
&= 2\cdot 2^k + 2\underbrace{(-1)^{k-1}(-1)^2}_{(-1)^{k+1}} - (-1)^{k+1} \\
&= 2^{k+1} + (-1)^{k+1}, \text{ as wanted}
\end{aligned}
$$

$\therefore$ By induction, $b_n = 2^n + (-1)^n \ \forall n \geq 0$

24. [2] When Christi and Gary go out for dinner it is either just the two of them, or the two of them together with one or both of their two closest friends. This term they have gone out with each of these friends a total of 7 times, and with both of them together 3 times. If, over the term, Christi and Gary have gone out for dinner a total of 18 times, how many times have the two of them gone out for dinner together with neither of their closest friends?

A: set of times they go out with friend 1
B: " " " " " " " friend 2



They have gone out with neither of their closest friends 7 times.

25. [2] Let $A = \{1, 2, 3, 4, 5, 6\}$. Fill in each blank. No justification is necessary.

(a) $|A \times A| =$ $6 \times 6 = 36$.

(b) The number of subsets of A that contain none of 1, 2, and 3 is $1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 = 8$.

(c) The number of functions $f : A \to \{w, x, y, z\}$ where $f(1) = w$, $f(2) = z$, $f(5) = x$, and $f(6) \neq y$ is $1 \cdot 1 \cdot 4 \cdot 4 \cdot 1 \cdot 3 = 48$.

(d) The number of relations on A is 2^{36}.

/END