

UNIVERSITY OF VICTORIA
DECEMBER EXAMINATIONS 2016

MATH 122: Logic and Foundations

CRN: 12190 [A01], 12191 [A02], 12192 [A03], 14105 [A04]

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NAME: _____

V00#: _____

Duration: 3 Hours.

Answers are to be written on the exam paper.

No calculator is necessary, but a Sharp EL-510R or a Sharp EL-510RNB calculator is allowed.

This exam consists of 27 questions, for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

4. [4] Use known logical equivalences and inference rules to show that the following argument is valid.

$$\frac{\begin{array}{l} \neg(q \wedge p) \\ \neg r \rightarrow p \\ q \end{array}}{\therefore r}$$

5. [3] Write the following argument in symbolic form, and then give a counterexample to show that it is invalid. Remember to define the letters you use for statements!

Either I wear a red tie or I wear blue socks

I am wearing blue socks

I am not wearing a red tie

6. [2] Let A , B , and C be sets. Use the blank to indicate whether each statement is True or False. No justification is necessary.

___ There are exactly three different sets X such that $\{1, 3\} \subseteq X \subsetneq \{1, 2, 3, 4\}$.

___ If some element of A is an element of B , then $A \subseteq B$.

___ If $A \subseteq C$ and $B \subseteq C$ then $A \not\subseteq B$.

___ If $A \times B = A \times C$, then $B = C$.

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7. [4] Suppose that $A \cup B = B$. Prove that $B^c \subseteq A^c$, using an argument that starts with “Take any $x \in B^c \dots$ ”.
8. [4] Let A, B and C be sets. Prove that $A \setminus (B \setminus C) = (A \setminus B) \cup (A \setminus C^c)$.
9. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.
- ___ Any relation on $\{1, 2, 3\}$ that contains at least 3 ordered pairs is reflexive.
 - ___ The relation $\mathcal{R} = \{(1, 1), (2, 2)\}$ on \mathbb{N} is symmetric, antisymmetric and transitive.
 - ___ If \mathcal{R} is relation on a set $\{1, 2, 3, 4, 5\}$ that is both symmetric and antisymmetric, then \mathcal{R} contains at most 5 ordered pairs.
 - ___ If \mathcal{R} and \mathcal{S} are transitive relations on a set A , then $\mathcal{R} \cap \mathcal{S}$ is a transitive relation on A .

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10. [2] Give an example of antisymmetric relations \mathcal{R} and \mathcal{S} on $A = \{1, 2, 3\}$ such that $\mathcal{R} \cup \mathcal{S}$ is not antisymmetric.
11. [4] Let \sim be the relation on the set of all subsets of $\{1, 2, 3, 4\}$ by $X \sim Y \Leftrightarrow$ the smallest element of X equals the smallest element of Y . Prove that \sim is an equivalence relation and find the equivalence class $[\{2, 4\}]$.

12. Let A and B be sets. Complete the following definitions.

(a) [1] A function $f : A \rightarrow B$ is:

(b) [1] A function $f : A \rightarrow B$ is 1-1 if:

(c) [1] A function $f : A \rightarrow B$ is onto if:

13. (a) [4] Suppose a and b are real numbers such that $a < b$. Let $f : (0, 1) \rightarrow (a, b)$ be defined by $f(x) = (b - a)x + a$. Prove that f is invertible.

(b) [1] Find a formula for f^{-1} .

14. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

___ If $f : A \rightarrow B$ is a function, then B is the range of f .

___ $f \circ g = g \circ f$ for all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$.

___ There exists a 1-1 function $f : \mathbb{Z} \rightarrow \mathbb{N}$.

___ If $f : A \rightarrow B$ and $g : B \rightarrow C$ are both onto, then $g \circ f$ is onto.

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15. [2] Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$. Suppose that $f : A \rightarrow B$ is $f = \{(1, e), (2, b), (3, a), (4, d)\}$. How many different functions $g : B \rightarrow A$ are such that $g \circ f = \iota_A$?
16. [4] Use the Euclidean Algorithm to find $d = \gcd(3142, 900)$ and then use your work to find integers x and y such that $3142x + 900y = d$.
17. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.
- When -120 is divided by -11 , the remainder is -1 .
 - For all integers n , the integers n and $n + 1$ are relatively prime.
 - If a and b are integers such that $\text{lcm}(a, b) < |ab|$, then $\gcd(a, b) > 1$.
 - If a and d are positive integers such that $d \mid a$, then there exists an integer n such that $\gcd(a, n) = d$.

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18. [1] Complete this definition. Let a and b be integers. We say that a divides b , and write $a \mid b$, if:
19. [3] Let a, b and d be integers such that $d \mid ab$ and $\gcd(a, d) = 1$. Prove that $d \mid b$. Give an example to show that the conclusion may not hold if $\gcd(a, d) > 1$.
20. [3] Find the base 9 representation of 2016.
21. [2] Use the blank to indicate whether each statement is True or False. All variables are integers. No justification is necessary.
- ___ The last digit of 7^{201} is 7.
 - ___ There exists an integer k such that $7k \mid (144)^5$
 - ___ If $a \equiv 40 \cdot 38^9 - 5 \pmod{13}$, then $13 \mid a^2 - 10$.
 - ___ $(11110101)_2 = (F5)_{16}$.

22. [3] Let n be a positive integer. Prove that

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = n.$$

Hint: consider the cases of n even and n odd.

23. [4] Let a_0, a_1, a_2, \dots be the sequence defined by $a_0 = 2$, and $a_n = 3a_{n-1} + 2$ for $n \geq 1$. Find a_1, a_2, a_3 and a_4 , then use your work to obtain a formula for a_n (note: a formula, not just a summation). It is not necessary prove that your formula is correct.

24. [4] Use induction to prove that, for all $n \geq 1$,

$$1(2) + 2(3) + 3(4) + \cdots + n(n+1) = n(n+1)(n+2)/3.$$

25. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

___ If $A \times A$ has 100 elements, then A has 10 elements.

___ There are $2^{(n^2)}$ relations on a set with n elements.

___ If A has two elements, then there are no transitive relations on A .

___ A set with 7 elements has $2^7 - 1$ different non-empty proper subsets.

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26. [4] Let b_0, b_1, \dots be the sequence defined by $b_0 = 1$, $b_1 = 4$ and $b_n = 4b_{n-1} - 4b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = (n + 1)2^n$ for all $n \geq 0$.

27. [2] A bicycle shop surveyed 50 customers and found that 30 own a mountain bike, 35 own a road bike, and 15 did not currently own a bicycle. How many of the 50 customers own both a mountain bike and a road bike?

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