UNIVERSITY OF VICTORIA
DECEMBER EXAMINATIONS 2017

MATH 122: Logic and Foundations

CRN: 12188 [A01], 12189 [A02], 12191 [A04]

Please circle your instructor's name.

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NAME: ______________________________

V00#: ______________________________

Duration: 3 Hours.

Answers are to be written on the exam paper.
No calculator is necessary, but a Sharp EL-510R or a Sharp EL-510RN calculator is allowed.

This exam consists of 26 questions, for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [3] Use a truth table to determine whether $(p \leftrightarrow \neg q) \rightarrow (\neg p \vee \neg q)$ is a tautology. (Here, and elsewhere, be sure to clearly state your answer to the question!)

2. [4] Use known logical equivalences to show that $\neg[p \vee (\neg p \wedge q)]$ is logically equivalent to $\neg[p \vee q]$.

3. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

___ The contrapositive of $p \rightarrow \neg q$ is $\neg p \rightarrow q$.

___ If the statement $(p \vee q) \wedge (\neg r \vee q)$ is true, then q must be true.

___ If $p \rightarrow q$ is true, then its converse is false.

___ For the universe of integers, the statements $\forall x, \exists y, x = -y$ and $\exists y, \forall x, x = -y$ are both true.

4. [4] Use known logical equivalences and rules of inference to show the given argument is valid. Justify each step.

$$\begin{array}{rc} & q \to p \\ & \neg(r \wedge p) \\ & r \\ \hline \therefore & \neg q \end{array}$$

5. [3] Give a counterexample to show the following argument is invalid.

$$\begin{array}{rc} & p \to r \\ & p \vee \neg q \\ \hline \therefore & \neg r \end{array}$$

6. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

___ If the integer n^2 is a multiple of 3, then n is a multiple of 3.

___ If $bk_1 + r_1 = bk_2 + r_2$, where $0 \leq r_1 < |b|$ and $0 \leq r_2 < |b|$, then $k_1 = k_2$ and $r_1 = r_2$.

___ The Fundamental Theorem of Arithmetic states that there are infinitely many prime numbers.

___ If the integer $n > 1$ has no prime divisor which is less than or equal to $\sqrt{n}$, then n is prime.

7. [4] Let a, b, c be integers. Prove that if $c \mid a$ and $c \mid (a+b)$, then $c \mid b$.

8. [3] Find the base 12 representation of 1596. Use T and E to represent the digits 10 and 11, if necessary.

9. [2] Use the blank to indicate whether each statement is True (T) or False (F). All variables are integers. No justification is necessary.

   ___ For all $a, b \in \mathbb{Z}$, if $gcd(a, b) = 1$ then $lcm(a, b) = ab$.

   ___ $321 \equiv -123 \pmod{12}$.

   ___ Let $d_k d_{k-1} \ldots d_0$ be the base 10 representation of n.
   Then $n \equiv d_k + d_{k-1} + \cdots + d_0 \pmod 3$

   ___ The last digit in the base 10 representation of 122^{122} is 8.

10. [4] Use the Euclidean Algorithm to find $d = gcd(630, 196)$ and then use your work to find integers x and y such that $630x + 196y = d$.

11. [3] Let $a = 6!$ and $b = 10!$. Find the prime decomposition of a, and of b, and then use these to find $lcm(a, b)$.

12. [3] Let p be a prime number, and $a, b \in \mathbb{Z}$. Suppose that $p|ab$, but p does not divide b. Prove, or explain why, $p|a$. Give an example to show that the statement can be false if p is not prime.

13. [4] Use induction to prove that $(n+1)! > 3^n$ for all integers $n \geq 4$.

14. Let $a_0, a_1, \ldots$ be the sequence recursively defined by $a_0 = 2$, and $a_n = 7a_{n-1} + 2$ for $n \geq 1$.

    (a) [2] Find a_1, a_2, a_3, and a_4. Leave each of these as a sum rather than computing a numerical value.

    (b) [2] Based on your work in part (a), guess a formula for a_n, for all integers $n \geq 0$. Give your final answer as a closed-form formula that is not a summation.

15. [4] Let $a_1, a_2, \ldots$ be the sequence recursively defined by $a_1 = 3, a_2 = 7$, and $a_n = 5a_{n-1} - 6a_{n-2}$, for $n \geq 3$. Use (the strong form of) induction to prove that $a_n = 2^n + 3^{n-1}$ for all $n \geq 1$.

16. [3] Two sets A and B have elements in a universe with $|\mathcal{U}| = 140$. If $|A \cup B| = 70$, $|A| = 50$, and $|B^c| = 82$. How many elements are in exactly one of the two sets?

17. [4] Let A, B, C be sets. Prove that if $A \subseteq B$, then $A \times C \subseteq B \times C$.

18. [3] Give a counterexample to show the following statement is false: "*For any sets A, B, C, if $A \cap B = A \cap C$ then $B = C$.*"

19. [2] Let $A = \{1, \{2, 3\}, \{1, 3\}\}$ and $B = \{1, 2, 3, \{1, 2, 3\}\}$. Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

___ $A \subseteq B$.

___ $\emptyset \subseteq B$.

___ $2 \in A$.

___ $|A \cup B| = 6$.

20. Let $\mathcal{R}$ be the relation on $\mathbb{Z}$ defined by $(x, y) \in \mathcal{R} \Leftrightarrow x + y$ is even.

    (a) [3] Prove that $\mathcal{R}$ is an equivalence relation.

    (b) [1] Find, or describe, the equivalence class [122].

21. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

    ___ There are $2^{(2^4)}$ relations on $\{1, 2, 3, 4\}$

    ___ The relation $\mathcal{R} = \{(1, 1), (1, 2), (3, 1), (3, 2)\}$ on $\{1, 2, 3\}$ is anti-symmetric.

    ___ For any non-empty set A, a function $f : \{1, 2, 3\} \to A$ contains exactly 3 ordered pairs.

    ___ There is no 1-1 function from $\mathbb{Z}$ to $\{1, 2, \ldots\}$.

22. [3] Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = \lceil n/3 \rceil$. Prove that f is onto.

23. Let $A = \{x, y, z\}$, $B = \{a, b, c, d\}$, and $C = \{u, v, w\}$. Let $f :\ A \to B$ and $h :\ A \to C$ be $f = \{(x, c), (y, b), (z, d)\}$, $\quad h = \{(x, v), (y, w), (z, u)\}$.

   (a) [1] Find a function $g :\ B \to C$ such that $g \circ f = h$. Write g as a set of ordered pairs.

   (b) [1] Write $f \circ h^{-1}$ as a set of ordered pairs.

24. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

   ___ The function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = 2x$ has an inverse.

   ___ The identity function $\iota : A \to A$ is an equivalence relation on A.

   ___ If there exists a 1-1 and onto function $f : A \to B$, then $|A| = |B|$.

   ___ If $a, b \in \mathbb{R}$ and $a < b$, then there is a bijection (i.e. a 1-1 correspondence) from $(0, 1)$ to (a, b).

25. Consider the set $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{1, 2, \ldots\}$.

    (a) [1] Let $k \geq 2$ be an integer. How many ordered pairs $(a, b) \in \mathbb{N} \times \mathbb{N}$ are such that $a + b = k$?

    (b) [1] Using part (a) or some other method, describe a sequence that contains every element of $\mathbb{N} \times \mathbb{N}$ exactly once.

    (c) [2] Explain why part (b) proves that $\mathbb{N} \times \mathbb{N}$ is a countably infinite set.

26. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

    ___ The set of prime numbers does not have the same cardinality as the set of rational numbers.

    ___ The open interval $(0, 1)$ does not have the same cardinality as $\mathbb{R}$.

    ___ If X is an uncountable set and $Y \subseteq X$, then Y is uncountable.

    ___ If A, B, C are countable sets, then there exists a sequence containing every element of $A \cup B \cup C$ exactly once.

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