UNIVERSITY OF VICTORIA
DECEMBER EXAMINATIONS 2018

MATH 122: Logic and Foundations

CRN: 12202 (A01), 12203 (A02), and 12205 (A04)
Please circle your section number

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NAME: ______________________________

V00#: ______________________________

Duration: 3 Hours.

Answers are to be written on the exam paper.

No calculator is necessary, but a Sharp EL-510R, Sharp EL-510RNB, or Sharp EL-510RTB calculator is allowed.

This exam consists of 29 questions, for a total of 80 marks. Unless otherwise specified, in order to receive full or partial credit, you must show your work and justify your answers.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [3] Suppose the statement $p \leftrightarrow \neg q$ is true. Find all combinations of truth values for p, q, and r such that the statement $(r \leftrightarrow p) \wedge \neg(r \vee \neg q)$ is true.

2. [3] Use known logical equivalences to show that $(q \to p) \to \neg q$ is logically equivalent to $\neg(p \wedge q)$.

3. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

   ___ The contrapositive of the statement "*if this question has the answer false, then the next question has the answer true*" is "*if the next question has the answer false, then this question has the answer true*".

   ___ The converse of the statement $p \to \neg q$ is $\neg p \to q$.

   ___ The statement $\forall x, (x^2 = 3) \to (x > 10)$ is true for the universe $\mathbb{Q}$.

   ___ The negation of the statement $\exists x, \exists y, xy = \pi$ is $\forall x, \exists y, xy \neq \pi$.

4. [3] Use rules of inference and known logical equivalences to show that the following argument is valid.

$$\begin{array}{rc} & q \vee r \\ & \neg(q \wedge \neg p) \\ & \neg r \\ \hline \therefore & p \end{array}$$

5. [3] Give a counterexample to show that the statement "*For all sets A, B, and C, if $A \setminus C = B \setminus C$, then $A = B$.*" is false. Remember to explain why your example demonstrates that the statement is false.

6. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. The letters p, q, and r represent statements.

   ___ If $p \Rightarrow q$, $q \Rightarrow r$, and $r \Rightarrow p$, then p, q, and r are all logically equivalent.

   ___ If the conclusion of an argument can never be true, then the argument is not valid.

   ___ If q is a tautology and p is any statement, then $p \Rightarrow q$.

   ___ If $s(x)$ is an open statement and the universe of x is not empty, then $[\forall x, s(x)] \Rightarrow [\exists x, s(x)]$.

7. [3] Let A and B be sets. Suppose that $A \subseteq B$. Prove that $A \cup B = B$.

8. [3] Let A and B be sets. Show that $(A^c \cap B)^c \cap (A^c \setminus B)^c = A$. Hint: set-theoretic identities.

9. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. The letters A, B, and C represent sets.

___ If the symmetric difference $A \oplus B \neq \emptyset$, then A is not a subset of B.

___ $\mathcal{P}(\{1,2\}) = \{\{1\}, \{2\}, \{1,2\}\}$.

___ If $A \subsetneq B$, then $A^c \subsetneq B^c$.

___ $(A \setminus B) \cup C = (A \setminus C) \cup (B \setminus C)$.

10. [4] Use induction to prove that any integer $n \geq 12$ can be written as a sum of 3s and 7s. (Note: a sum that involves only threes is acceptable, and the same for sevens.)

11. [3] Let $a_0, a_1, \ldots$ be the sequence defined by $a_0 = 9$, and $a_n = 6a_{n-1} + 9$ for $n \geq 1$. Find a_1, a_2, a_3 and a_4 – there is no need to simplify – then use your work to guess a formula for a_n. It is not necessary to prove that your formula is correct, but note that it is not sufficient to express a_n as a sum.

12. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. Unless otherwise specified, lower case letters represent integers.

    ___ If $s(n)$ is an open statement such that $s(k)$ logically implies $s(k+1)$ for all $k \geq 0$, then $s(n)$ is true for all $n \geq 0$.

    ___ There is an integer x such that $(1x)_{10} = (x1)_4$.

    ___ Whether a number is even or odd depends on the base in which it is represented.

    ___ If $p_1, p_2, \ldots, p_n$ are odd prime numbers, then the number $N = p_1 p_2 \cdots p_n + 2$ is not divisible by any of $p_1, p_2, \ldots, p_n$.

13. [4] Let $b_0, b_1, \ldots$ be defined by $b_0 = 0,\ b_1 = 2$, and $b_n = 4b_{n-1} - 4b_{n-2},\ n \geq 2$. Use induction to prove that $b_n = n2^n$ for all $n \geq 0$.

14. [3] Let a, b, and c be integers such that $3a \,|\, b$ and $9b \,|\, c$. Prove that $27a \,|\, c$.

15. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. All lower case letters denote integers.

___ If $m > 1$ and $a = km + b$, then $a \equiv b \pmod{m}$.

___ $(2233)_4 = (AF)_{16}$

___ If $lcm(a, b) < ab$, then a and b are not relatively prime.

___ If $b > 1$ is even, then a number written in base b is even if and only if its last digit is even.

16. [3] Use the Euclidean Algorithm to find $d = gcd(2200, 572)$, and then use your work to find integers x and y such that $2200x + 572y = d$.

17. Let $a, b, c, p \in \mathbb{Z}$.

    (a) [3] Prove that if p is prime and $p|ab$, then $p|a$ or $p|b$.

    (b) [1] Give an example to show that the statement *if* $a|bc$, *then* $a|b$ *or* $a|c$ is not true for all integers a, b and c.

18. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. All lower case integers denote integers.

    ___ $gcd(2^{50}5^{30}7^{10}, 60 \times 77) = 2^2 5^1 7^1$.

    ___ If $a|b$ then $lcm(a, b) = |b|$.

    ___ $125 \cdot 7^4 - 2018 \equiv -11 \pmod 7$.

    ___ If $a \equiv 9 \pmod{12}$, then $a \equiv 3 \pmod 4$.

19. (a) [2] Find the last digit of 33^{123}.

(b) [1] Find the last digit of the base 3 representation of 33^{123}.

20. [3] Let A, B, C, D be sets such that $A \subseteq C$ and $B \subseteq D$. Prove that $A \times B \subseteq C \times D$.

21. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

___ For any set A, the relation $\mathcal{R} = \emptyset$ is a symmetric, transitive, and antisymmetric relation on A.

___ If $|A| = n$, and $\mathcal{R}$ is a relation on A that contains fewer than n ordered pairs, then $\mathcal{R}$ is not reflexive.

___ If $|A| = n$, then there are 2^{n^2} relations on A.

___ If $\mathcal{R}$ a symmetric and transitive relation on the set A, then $\mathcal{R}$ is reflexive.

22. [3] Let $\mathcal{R}$ be the relation on $\mathbb{N} = \{1, 2, \ldots\}$ where $(a, b) \in \mathcal{R}$ if and only if $\frac{a}{b} \leq \frac{b}{a}$. Prove that $\mathcal{R}$ is reflexive and antisymmetric.

23. [2] Let $\mathcal{R}$ be an equivalence relation on $A = \{1, 2, 3, 4, 5, 6\}$. Suppose $\mathcal{R}$ has exactly three different equivalence classes, and $(1, 3), (1, 4), (6, 2) \in \mathcal{R}$. Find the partition of A into equivalence classes.

24. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

    ___ For any set A, the identity function ι_A is a reflexive relation on A.

    ___ If A and B are sets with $|A| = 5$ and $|B| = 7$, then a function $f : A \to B$ contains exactly 7 ordered pairs.

    ___ If $f : A \to B$ is 1-1, then $|A| \geq |B|$.

    ___ There are $2^4 - 2$ onto functions $f : \{1, 2, 3, 4\} \to \{0, 1\}$.

25. Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^2 + 5$.

    (a) [2] Show that f is 1-1.

    (b) [1] Show that $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2 + 5$, is not 1-1.

26. Suppose $f : A \to B$ and $g : B \to C$ are both onto functions.

    (a) [3] Prove that $g \circ f : A \to C$ is an onto function.

    (b) [1] Suppose f and g are also both 1-1. Is $g \circ f$ invertible? Explain.

27. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

    ___ Every subset of an uncountable set is uncountable.

    ___ If A and B are countable sets, then $A \cup B$ is countable.

    ___ The relation $\mathcal{R}$ on the set of all subsets of the universe defined by $(X, Y) \in \mathcal{R}$ if and only if there is a 1-1 correspondence between X and Y is an equivalence relation.

    ___ Any interval of real numbers with positive length has the same cardinality as $\mathbb{R}$.

28. [3] Use a diagonal sweeping argument to prove that the set $\mathbb{Z} \times \mathbb{Z}$ is countable.

29. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

___ If $|A| = n$, then the number of non-empty subsets of A is $2^n - 1$.

___ The number of 1-1, onto functions $f : \{1, 2, 3\} \to \{1, 2, 3\}$ is the same as the number of 1-1 functions $g : \{1, 2, 3\} \to \{1, 2, 3\}$.

___ The number of subsets of $\{a, b, c, d, e\}$ that contain a or b is $2^5 - 2^3$.

___ $|\mathcal{P}(X)| > |X|$ for any set X.

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