MATH 122 – Logic and Foundations **Final Exam**

Date: Dec. 13, 2023 **Time Limit: 3 hours**

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Before starting your test, enter your name and student ID (clearly) on this page.

This exam contains 22 pages (including this one, and scrap pages)
and 25 questions, worth a total of 80 marks.

You may not open the exam until instructed to do so. In the meantime, please read the following instructions:

- No textbooks or class notes are allowed on this exam. A calculator is not necessary, but a Sharp EL-510R, EL-510RN, or EL-510RNB calculator is allowed.
- Answer each question in the space immediately below that question. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

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[3] 1. Find all possible truth values of p, q, and r such that $(p \to q) \to \neg r$ is false.

[4] 2. Use known logical equivalences to show that $p \to (q \vee r)$ is logically equivalent to $(p \wedge \neg r) \to q$.

[2] 3. Check the appropriate circle to indicate whether each statement is true or false. You do not need to provide any reasoning.

	True	**False**	
(a)	○	○	If $(\neg p) \Rightarrow q$ then $p \Rightarrow (\neg q)$.
(b)	○	○	The negation of $\forall x, \exists y, x \vee y$ is $\exists y, \forall x, (\neg x) \wedge (\neg y)$.
(c)	○	○	The statement $\forall x,\ x \leq 3x$ is true for the universe of positive real numbers.
(d)	○	○	If an argument is valid, then its premises are all true.

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[4] 4. Use known logical equivalences and inference rules to show that the following argument is valid.

$$\begin{array}{r} (\neg q) \to p \\ (\neg r) \to s \\ (\neg q) \wedge u \\ \neg s \\ \hline \therefore p \wedge r \end{array}$$

[3] 5. Give a counterexample to show that the following argument is invalid.

$$\begin{array}{ll} & p \to q \\ & (\neg q) \to (\neg s) \\ & r \to s \\ \hline \therefore & p \end{array}$$

[2] 6. Let A, B, and C be sets. Check the appropriate circle to indicate whether each statement is true or false. You do not need to provide any reasoning.

	True	**False**	
(a)	◯	◯	If $A \subseteq B$, then $A^c \subseteq B^c$.
(b)	◯	◯	If $A \subseteq \mathbb{Z}$ and $B \subseteq \mathbb{Q}$, then $A \subseteq B$.
(c)	◯	◯	If $10 \in A$ and $3 \in B$, then $30 \in A \times B$.
(d)	◯	◯	If $C \in \mathcal{P}(A)$, then $A \cap C = C$.

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[4] 7. Let A and B be sets. Show that $(A \setminus B^c) \cup (A^c \cup B)^c = A$.
Hint: Use set-theoretic identities.

8. Let $A \subseteq B$.

[3] (a) Prove that $A \setminus B = \emptyset$.

[1] (b) Prove that $A = A \cap B$.

[2] 9. Let $A = \{1, 2, 3\}$. Fill in each blank. No justification is necessary.

(a) The number of subsets of A that contain 1 but not 2 is ____________________.

(b) The number of non-empty proper subsets of A is ____________________.

(c) The number of onto (surjective) functions $f : A \to \{w, x, y, z\}$ is ____________________.

(d) The number of relations on A is ____________________.

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[2] 10. The three most common colours in the 193 of the member nations of the United Nations are red, white, and blue. There are 52 flags that contain all three colours, 103 that contain both red and white, 66 that contain both red and blue, 73 that contain both white and blue, 145 that contain red, 132 that contain white, and 104 that contain blue. How many flags contain exactly two of the three colours?

[2] 11. Check the appropriate circle to indicate whether each statement is true or false. You do not need to provide any reasoning. All variables are integers.

	True	False	
(a)	○	○	If $a \mid b$ and $b \mid a$, then $a = b$.
(b)	○	○	If $d \mid (a+b)$, then $d \mid a$ and $d \mid b$.
(c)	○	○	The last digit of 3^{51} is 3.
(d)	○	○	If $a \equiv 0 \pmod 3$, then $a \not\equiv 0 \pmod 2$.

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[4] 12. Use the Euclidean Algorithm to find $d = \gcd(222, 207)$ and then use your work to find integers x and y such that $222x + 207y = d$.

[4] 13. Let a and b be positive integers. Prove that if $\gcd(a, b) = \operatorname{lcm}(a, b)$, then $a = b$.

[2] 14. Check the appropriate circle to indicate whether each statement is true or false. You do not need to provide any reasoning. All variables are integers.

	True	**False**	
(a)	◯	◯	If $a = 2^3 \cdot 3^7 \cdot 17^2$ and $b = 2^6 \cdot 3^2 \cdot 19^3$, then $\operatorname{lcm}(a, b) = 2^6 \cdot 3^7 \cdot 17^2 \cdot 19^3$.
(b)	◯	◯	If $\gcd(a, b) = 2$, then there exist integers x and y such that $ax + by = 10$.
(c)	◯	◯	If $\gcd(a, b) = p$ a prime, then $\gcd(a^3, b) = p$.
(d)	◯	◯	Let p be a prime. If $p \mid a$ then $p^k \mid a^k$ for any integer $k \geq 1$.

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[4] 15. Let $n = (d_3 d_2 d_1 d_0)_7$. Prove that $6 \mid n$ if and only if $6 \mid (d_3 + d_2 + d_1 + d_0)$.

[4] 16. Use induction to prove that $1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$ for all integers $n \geq 1$, where $n! = n(n-1)\cdots 2 \cdot 1$.

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[4] 17. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by $a_0 = 4$, and $a_n = a_{n-1} + 4n$ for $n \geq 1$. Find a_1, a_2, a_3, and a_4, then use your work to obtain a formula for a_n (note: a formula, not just a summation). It is not necessary prove that your formula is correct.

[4] 18. Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 6$, $b_1 = 23$ and $b_n = 7b_{n-1} - 12b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = 3^n + 5 \cdot 4^n$ for all $n \geq 0$.

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[4] 19. Let $f : \mathbb{R} \to (-2, \infty)$ be defined by $f(x) = x^2 - 2$. Prove that f is onto (surjective) but not one-to-one (injective).

[2] 20. Check the appropriate circle to indicate whether each statement is true or false. You do not need to provide any reasoning.

	True	False	
(a)	○	○	For any integer x, $\left\lfloor \frac{2x+1}{2} \right\rfloor = \left\lceil \frac{2x+1}{2} \right\rceil - 1$.
(b)	○	○	There exists an onto (surjective) function from $\{1, 2, 3\}$ to $\{1, 2, 3, 4, 5\}$.
(c)	○	○	A function $f : A \to B$ is one-to-one (injective) if for every $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in f$.
(d)	○	○	A function $f : A \to B$ has an inverse if and only if the relation g from B to A defined by $g = \{(b, a) : (a, b) \in f\}$ is a function.

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[4] 21. For sets A, B, and C let $f : A \to B$ and $g : B \to C$ be one-to-one (injective) functions. Prove that $g \circ f : A \to C$ is one-to-one (injective).

[5] 22. Let $A = \{10, 11, 12, 13, \ldots, 99\}$, and let $\mathcal{R}$ be the relation on A defined by $(x, y) \in \mathcal{R}$ if and only if the sum of the digits of x equals the sum of the digits of y. For example $15\,\mathcal{R}\,24$ (i.e. $(15, 24) \in \mathcal{R}$) since $1 + 5 = 2 + 4$.

Prove that $\mathcal{R}$ is an equivalence relation on A. How many distinct equivalence classes are there determined by $\mathcal{R}$? Explain.

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[2] 23. Check the appropriate circle to indicate whether each statement is true or false. You do not need to provide any reasoning.

	True	False	
(a)	◯	◯	Suppose A is a set with $\|A\| = n$. There exists a reflexive relation $\mathcal{R}$ on A that contains fewer than n ordered pairs.
(b)	◯	◯	If $\mathcal{R}$ is a symmetric relation on $\{1, 2, 3, 4\}$ and $(4, 1) \in \mathcal{R}$, then $\mathcal{R}$ is antisymmetric.
(c)	◯	◯	If $\mathcal{R}$ is a reflexive relation on $\{1, 2, 3\}$ with $(1, 2) \in \mathcal{R}$, then $\mathcal{R}$ is not a function.
(d)	◯	◯	If $\mathcal{R}$ is an antisymmetric and transitive relation on $\{1, 2, 3, 4\}$ where $(3, 1) \in \mathcal{R}$ and $(4, 1) \in \mathcal{R}$, then $(3, 4) \in \mathcal{R}$.

[3] 24. Let A be a countably infinite set. Using a diagonal sweeping argument, prove that $A \times \mathbb{N}$ is countable.

[2] 25. Check the appropriate circle to indicate whether each statement is true or false. You do not need to provide any reasoning.

	True	False	
(a)	◯	◯	If A is an uncountable set, and $B \subseteq A$, then B is uncountable.
(b)	◯	◯	For any real number b such that $b > 1$, the open interval $(1, b)$ is uncountable.
(c)	◯	◯	If A is a countable set, then $A \cup \{-1, 1\}$ is countable.
(d)	◯	◯	The set $\{\ldots, -16, -9, -4, -1, 0, 1, 4, 9, 16, \ldots\}$ is uncountable.

[END OF EXAM]

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