UNIVERSITY OF VICTORIA
EXAMINATIONS DECEMBER 2025

MATH 122: Logic and Foundations

CRN: 12092 (A01), 12093 (A02), 12094 (A03), 12095 (A04)

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Duration: 3 Hours.

Answers are to be written on the exam paper.
No calculator is necessary, but a Sharp EL-510R (plus some letters) calculator is allowed.

This exam consists of 21 questions for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed. Question 1 consists of 28 true/false questions labelled TF 1 to TF 28 to be answered on the bubble sheet at the end of the booklet. True is A and False is B on the bubble sheet.

There are 12 numbered pages, not including covers, and one bubble sheet.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

Do your rough work here. Nothing written here will be marked.

1. [14] Use the **bubble sheet** provided on the last page of the test booklet to indicate whether each statement is **True (A)** or **False (B)**. No justification is necessary.

[TF 1] $p \to r \Leftrightarrow \neg r \vee p$.

[TF 2] If s_1 is a tautology, then $s_1 \to s_2$ is true for every statement s_2.

[TF 3] $p \vee p$ logically implies $p \wedge p$.

[TF 4] Let the universe be the real numbers. The negation of "*For all x there exists y such that $x + y = 0$*" is "*For all x there does not exist y such that $x + y = 0$.*"

[TF 5] If $n \in \mathbb{N}$, then $1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$.

[TF 6] There exists an integer n such that $3^{9999}n = 7^{444}$.

[TF 7] The Principle of Inclusion-Exclusion implies that $|A \cup B| = |A \oplus B| - |A \cap B|$ for any two finite sets A and B.

[TF 8] $\sqrt{2} \in \mathbb{R}$.

[TF 9] If a and b are positive integers, then a and b are relatively prime if and only if $lcm(a, b) = ab$.

[TF 10] $gcd(74, 4) = 2$.

[TF 11] $|\mathbb{Q} \cap [0, 1)| = |\mathbb{N}|$.

[TF 12] $77^{1234567} \equiv -9 \pmod 8$.

In the following four questions, let $X = \{\emptyset\}$

[TF 13] $|X| = 0$.

[TF 14] $|\mathcal{P}(X)| = 2$.

[TF 15] $X \subseteq \mathcal{P}(X)$.

[TF 16] $\emptyset \subseteq X$.

[TF 17] Every equivalence relation is antisymmetric.

[TF 18] The relation $\mathcal{R} = \{(1,1),(1,2),(2,1),(2,2),(3,4)\}$ on $\{1,2,3,4\}$ is transitive.

[TF 19] The relation $\sim$ on $\{2,3,4,\ldots\}$ where $a \sim b$ if and only if a and b are both prime or both composite is an equivalence relation.

[TF 20] The composition of two surjective functions is onto.

[TF 21] The relation $\{(A,B) : A \subseteq B\}$ on $\mathcal{P}(\{1,2,3\})$ is a function.

[TF 22] Every bijection has an inverse.

[TF 23] If $f(x) = 3x + 2$ and $g(x) = (x-5)^2$, then $f \circ g(x) = (3x-3)^2$.

[TF 24] The set $\{x : x \in \mathbb{R} \text{ and } x^2 - 2x + 1 = 0\}$ is uncountable.

[TF 25] A real number is irrational if and only if it can be written as a fraction of two integers in lowest terms.

[TF 26] $A \cap B \subseteq A \cup B$ for any two sets A and B.

[TF 27] If $A = B$, then $A \subseteq B$.

[TF 28] If $A \subseteq B$ and $B \subseteq A$, then $A \setminus B = B \setminus A$.

2. [3] Use any method to determine whether $\neg(\neg p \vee q) \to (\neg p \to q)$ is a contradiction or a tautology or neither. Write a sentence that explains how your work shows that your conclusion is correct.

3. [4] Use the Laws of Logic and known logical equivalences to show that $\neg(p \to (p \vee q)) \vee \neg q$ is logically equivalent to $\neg q$.

4. [4] Use the Laws of Logic, known logical equivalences and inference rules to show that the following argument is valid. Justify each step.

$$\begin{array}{ll} & p \to (q \to \neg s) \\ & \neg r \to s \\ & p \wedge q \\ \hline \therefore & r \end{array}$$

5. [3] Give a counterexample to show that the following argument is invalid. Be sure to explain why your counterexample shows that it is invalid.

$$\begin{array}{ll} & r \to p \\ & q \vee (\neg s \to s) \\ \hline \therefore & \neg p \to q \end{array}$$

6. [3] Let $a, b, c, d \in \mathbb{Z}$. Prove that if $a \mid b$ and $c \mid d$, then $ac \mid bd$.

7. [3] Find the base 12 representation of 6902. Use the symbols A, B to represent the digits 10 and 11, respectively.

8. [4] Use the Euclidean Algorithm to find $gcd(3852, 1584)$ and then use your work to find integers x and y such that $3852x + 1584y = gcd(3852, 1584)$.

9. [2] Let $n = (d_2d_1d_0)_{16}$ be a 3-digit number in hexadecimal. Prove that

$$n \equiv (d_2 + d_1 + d_0) \pmod{15}.$$

10. [3] Let p be a prime number and let a and b be integers. Prove that if $p \mid ab$, then $p \mid a$ or $p \mid b$.

11. [5] Use induction to prove that $2 + 4 + 6 + 8 + \cdots + 2n = n^2 + n$ for every integer $n \geq 1$.

12. [2] Let $b_0, b_1, \ldots$ be the sequence recursively defined $b_0 = 3$ and $b_n = -2b_{n-1} + 1$ for $n \geq 1$. Find b_1, b_2 and b_3.

13. [2] Give a recursive definition of the sequence $d_1, d_2, d_3, \ldots$ defined by $d_n = 1! + 2! + \cdots + n!$ for $n \geq 1$.

14. [2] In a group of 100 people, there are 60 who like artichokes and 70 who like bok choy and 20 who like neither. How many of the people like both?

15. [5] Let $a_0, a_1, \ldots$ be the sequence recursively defined by $a_0 = 0$, $a_1 = 2$ and $a_n = a_{n-1}+6a_{n-2}+6$ for all $n \geq 2$. Use induction to prove that $a_n = 3^n - 1$ for all $n \geq 0$.

16. [4] Let A, B and C be sets such that $A \subseteq B$. Prove that $A \cap C \subseteq B \cap C$. Your proof must begin with "Take any $x \in A \cap C \ldots$"

17. [4] Give a counterexample to show that the following statement is false: "*For any sets* A, B *and* C*, it holds that* $(A \oplus B) \oplus C = (A \cup B) \cup C$."

18. (a) [3] Let $\sim$ be the relation on $\mathbb{Z} \setminus \{0\}$ defined by $x \sim y$ if and only if $\frac{x}{y} \in \{-1, 1\}$. Prove that $\sim$ is an equivalence relation.

(b) [1] For the relation $\sim$ in part (a), write out all the elements of the equivalence class [77].

19. [2] Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = (n-2)^2$. Give a counterexample to prove that f is not 1-1.

20. [3] Prove that $A = \{n^2 : n \in \mathbb{N}\} \times \{\sqrt{n} : n \in \mathbb{N}\}$ is countable. *Hint*: Find a sequence $a_1, a_2, a_3, \ldots$ that includes each element of A at least once.

21. [4] Let $3\mathbb{N} = \{n \in \mathbb{N} : n \equiv 0 \pmod 3\}$. Find a bijection from $\mathbb{N}$ to $3\mathbb{N}$ and prove that it is a bijection.

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