UNIVERSITY OF VICTORIA
APRIL EXAMINATIONS 2018

MATH 122: Logic and Foundations

CRN: 21996 (A01), 21997 (A02), 21998 (A03), 21999 (A04)

Please circle your instructor and section:

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NAME: ______________________________

V00#: ______________________________

Duration: 3 Hours.

Answers are to be written on the exam paper.
No calculator is necessary, but a Sharp EL-510R or Sharp EL-510RN, or Sharp EL-510RNB calculator is allowed.

This exam consists of 25 questions, for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [3] Find all truth values for p, q and r for which $(\neg q) \wedge (r \rightarrow (\neg p))$ is true.

2. [4] Use known logical equivalences to show that $p \rightarrow (q \vee r)$ is logically equivalent to $(p \wedge (\neg r)) \rightarrow q$.

3. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

    ___ If $(\neg p) \Leftrightarrow q$ then $p \Leftrightarrow (\neg q)$.

    ___ The negation of $\forall x, \exists y, x \vee y$ is $\exists y, \forall x, (\neg x) \wedge (\neg y)$.

    ___ The statement $\forall x,\ x \leq 5x$ is true for the universe $\mathbb{R}$.

    ___ If an argument is valid, then its conclusion is true.

4. [4] Use known logical equivalences and inference rules to show that the following argument is valid.

$$\begin{array}{l} (\neg p) \to q \\ (\neg r) \to (\neg q) \\ \neg r \\ \hline \therefore\ p \end{array}$$

5. [3] Give a counterexample to show that the following argument is invalid.

$$\begin{array}{l} (\neg p) \to q \\ (\neg q) \to (\neg s) \\ r \to s \\ p \leftrightarrow (\neg r) \\ \hline \therefore\ p \end{array}$$

6. [2] Let A, B and C be sets. Use the blank to indicate whether each statement is True or False. No justification is necessary.

___ If $A \not\subseteq B$, then $B \subseteq A$.

___ If $A \subseteq \mathbb{Z}$ and $B \subseteq \mathbb{Q}$, then $A \subseteq B$.

___ If $10 \in A$ and $3 \in B$ then $30 \in A \times B$.

___ If $C \in \mathcal{P}(A)$ then $A \backslash C \in \mathcal{P}(A)$.

7. [4] Suppose that $A \subseteq B$. Prove that $A \subseteq A \cap B$, using an argument that starts with "*Take any* $x \in A$ ...". Are the sets A and $A \cap B$ actually equal? Explain.

8. [4] Let A and B be sets. Show that $(A \setminus B^c) \cup (A^c \cup B)^c = A$. Hint: set-theoretic identities.

9. [2] Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. Fill in each blank. No justification is necessary.

   (a) The number of subsets of A that contain neither 1 nor 2 is ________________.

   (b) The number of functions $f : A \to \{w, x, y, z\}$ where $f(1) = x$, $f(2) = w$ and $f(5) \neq x$, is ________________.

   (c) $|A \times A| =$ ________________.

   (d) The number of relations on A is ________________.

10. [2] An airline surveyed 200 passengers and recorded the following information about the type of non-alcoholic beverages they liked: 78 passengers liked coffee, 70 liked tea, and 66 liked orange juice; 35 liked both coffee and tea, 30 liked both tea and orange juice, 15 liked both coffee and orange juice, and 10 liked all three types of beverages. How many passengers liked only orange juice or none of the three beverages?

11. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. All variables are integers.

   ___ If $d \mid a$ and $d \mid b$, then $d \mid (b-a)$.

   ___ If $a \mid bc$, then $a \mid b$ or $a \mid c$.

   ___ $\mathrm{lcm}(2^4 7^6, 2^1 3^2 7^2) = 2^5 3^2 7^8$.

   ___ If $\gcd(a,b) = 5$ then there exist integers x and y such that $ax + by = 20$.

12. [4] Use the Euclidean Algorithm to find $d = gcd(2500, 1120)$ and then use your work to find integers x and y such that $2500x + 1120y = d$.

13. [4] Find the base 9 representation of 2018.

14. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. All variables are integers.

___ The integers 1000012^{66} and 122^{66} have the same last digit.

___ If $a \equiv 0 \pmod 3$ then $a \not\equiv 0 \pmod 2$.

___ If $ak = b$ then every prime divisor of a is a divisor of b.

___ $(110100)_2 = (C0)_{16}$.

15. [4] Let $n = (d_3d_2d_1d_0)_{10}$. Prove that $3 \mid n$ if and only if $3 \mid (d_3 + d_2 + d_1 + d_0)$.

16. [4] Use induction to prove that $1(2) + 2(3) + \cdots + n(n+1) = n(n+1)(n+2)/3$ for all integers $n \geq 1$

17. [4] Let $a_1, a_2, \ldots$ be the sequence defined by $a_1 = 6$, and $a_n = a_{n-1} + 6n$ for $n \geq 2$. Find a_2, a_3, a_4 and a_5, then use your work to obtain a formula for a_n (note: a formula, not just a summation). It is not necessary prove that your formula is correct.

18. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 2$, $b_1 = 5$ and $b_n = 5b_{n-1} - 6b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = 2^n + 3^n$ for all $n \geq 0$.

19. [4] Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 3x$. Prove that f is 1-1 but not onto.

20. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

    ___ For any $x \in \mathbb{R}$, $\lfloor -x \rfloor = -\lfloor x \rfloor$.

    ___ There exists a one-to-one function from $\{1, 2, 3, 4\}$ to $\{1, 2, 3\}$.

    ___ A function $f : A \to B$ is onto if for every element $a \in A$ there exists an element $b \in B$ such that $(a, b) \in f$.

    ___ A function $f : A \to B$ has an inverse if and only if it is 1-1 and onto.

21. [4] For sets A, B and C, let $f : A \to B$ and $g : B \to C$ be onto functions. Prove that $g \circ f : A \to C$ is onto.

22. [5] Let $A = \{1, 2, \ldots, 11\}$, and let $\sim$ be the relation on A defined by $a \sim b \Leftrightarrow 2 \mid (a+b)$. Prove that $\sim$ is an equivalence relation on A and find the partition of A it determines.

23. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

    ___ The relation $\{(1, 1)\}$ on $\{1, 2, 3\}$ is reflexive.

    ___ If $\mathcal{R}$ is an antisymmetric relation on $\{1, 2, 3, 4\}$ and $(1, 2) \in \mathcal{R}$ then $\mathcal{R}$ is not symmetric.

    ___ The relation $\sim$ on $\{2, 4, 6, \dots\}$ defined by $x \sim y$ if and only if $x - y = 2$ is transitive.

    ___ If $\sim$ is an equivalence relation on $\mathbb{Z}$ such that $1 \not\sim 3$, then $1 \not\sim 2$ or $2 \not\sim 3$.

24. [3] Let A and B be countably infinite sets. Prove that $A \cup B$ is countable.

25. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

    ___ If A is a countable set and B is an uncountable set, then $A \cup B$ is uncountable.

    ___ For any real numbers a, b such that $a < b$, the closed interval $[a, b]$ is uncountable.

    ___ If $A \subsetneqq B$ then $|A| < |B|$.

    ___ The set $\{1, 2, -3, -4, 5, 6, -7, -8 \dots\}$ is countable.

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