UNIVERSITY OF VICTORIA
APRIL EXAMINATIONS 2015

MATH 122: Logic and Foundations

CRN: 22014 (A01), 22015 (A02) and 24018 (A03)

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NAME: ____

V00#: ____

Duration: 3 Hours.

Answers are to be written on the exam paper.
No calculator is necessary, but a Sharp EL-510R or a Sharp EL-510RNB calculator is allowed.

This exam consists of 25 questions, for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [3] Find all truth values for p, q and r for which $q \wedge (\neg r \to p)$ is true.

2. [4] Use known logical equivalences to show that $p \to (q \vee r)$ is logically equivalent to $(p \wedge \neg q) \to r$.

3. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

   ___ If $p \Leftrightarrow q$ then $\neg p \Leftrightarrow \neg q$.

   ___ The negation of $\forall x, \exists y, x \wedge y$ is $\forall x, \exists y, \neg x \vee \neg y$.

   ___ The statement $\forall x,\ x \leq 2x$ is true for the universe $\mathbb{R}$.

   ___ If an argument is valid, then its conclusion is true.

4. [4] Use known logical equivalences and inference rules to show that the following argument is valid.

$$\begin{array}{rl} & p \to q \\ & r \to \neg q \\ & r \\ \hline \therefore & \neg p \end{array}$$

5. [3] Give a counterexample to show that the following argument is invalid.

$$\begin{array}{rl} & p \to q \\ & \neg q \to s \\ & r \to \neg s \\ & \neg p \leftrightarrow \neg r \\ \hline \therefore & \neg p \end{array}$$

6. [2] Let A, B and C be sets. Use the blank to indicate whether each statement is True or False. No justification is necessary.

___ If $A \times B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$.

___ If $A \subseteq \mathbb{Z}$ and $B \subseteq \mathbb{R}$, then $A \subseteq B$.

___ If $10 \in A$ and $3 \in B$ then $30 \in A \times B$.

___ If $C \in \mathcal{P}(A)$ then $A \backslash C \in \mathcal{P}(A)$.

7. [4] Suppose that $A \subseteq B$. Prove that $A \subseteq A \cap B$, using an argument that starts with "*Take any* $x \in A$ ...". Are the sets A and $A \cap B$ actually equal? Explain.

8. [4] Let A and B be sets. Show that $(A \setminus B) \cup (A^c \cup B^c)^c = A$. Hint: set-theoretic identities.

9. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

___ The relation $\emptyset$ on $\{1, 2, 3\}$ is reflexive.

___ If $\mathcal{R}$ is a symmetric relation on $\{1, 2, 3, 4\}$ and $(1, 2) \in \mathcal{R}$ then $\mathcal{R}$ is not antisymmetric.

___ The relation $\sim$ on $\{2, 4, 6, \dots\}$ defined by $x \sim y$ if and only if $x - y = 2$ is transitive.

___ If $\sim$ is an equivalence relation on $\mathbb{N}$ such that $1 \sim 2$ and $1 \not\sim 3$ then $2 \not\sim 3$.

10. [5] Let $A = \{1, 2, \ldots, 11\}$, and let $\sim$ be the relation on A defined by $a \sim b \Leftrightarrow 3 \mid (a-b)$. Prove that $\sim$ is an equivalence relation on A and find the partition of A it determines.

11. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

    ___ For any $x \in \mathbb{R}$, $\lceil -x \rceil = -\lceil x \rceil$.

    ___ There exists a one-to-one function from $\{1, 2, 3\}$ to $\{1, 2\}$.

    ___ All functions from a finite set A to a set B consist of the same number of ordered pairs.

    ___ A function $f : A \to B$ has an inverse if and only if it is a 1-1 correspondence.

12. [4] Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$. Prove that f is 1-1 but not onto.

13. [4] For sets A, B and C, let $f : A \to B$ and $g : B \to C$ be onto functions. Prove that $g \circ f : A \to C$ is onto.

14. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

    ___ If $p|a$ and $p|b$ then $p|a - b$.

    ___ For integers a, b and c, if $a|b$ and $b|c$ then $a|c$.

    ___ $\text{lcm}(2^3 7^4, 3^2 7^2) = 2^3 3^2 7^6$.

    ___ If $\gcd(a, b) = 3$ then there exist integers x and y such that $ax + by = 6$.

15. [4] Use the Euclidean Algorithm to find $d = gcd(2500, 1110)$ and then use your work to find integers x and y such that $2500x + 1110y = d$.

16. [4] Find the base 12 representation of 2015. Use the symbols δ and χ to represent 10 and 11, respectively.

17. [2] Use the blank to indicate whether each statement is True or False. All variables are integers. No justification is necessary.

    ___ The last digit of 2016^3 is 6.

    ___ If $a \equiv 0 \pmod 2$ then $a \not\equiv 0 \pmod 3$.

    ___ If $ak = b$ then every prime divisor of a is a divisor of b.

    ___ $(110010)_2 = (32)_{16}$.

18. [3] Suppose k has the prime factorization $p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$. Prove that if $s = k^2$, then every exponent in the prime factorization of s is even. Is the converse true? Explain.

19. [4] Use induction to prove that for all $n \geq 1$, $\frac{1}{1(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

20. [4] Let $a_1, a_2, \ldots$ be the sequence defined by $a_1 = 2$, and $a_n = a_{n-1} + 2n$ for $n \geq 2$. Find a_2, a_3, a_4 and a_5, then use your work to obtain a formula for a_n (note: a formula, not just a summation). It is not necessary prove that your formula is correct.

21. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 2,\ b_1 = 5$ and $b_n = 5b_{n-1} - 6b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = 2^n + 3^n$ for all $n \geq 0$.

22. [4] Let $\mathcal{F} = \{5, 10, 15, \ldots\}$. Use a diagonal sweeping argument to prove that $\mathcal{F} \times \mathcal{F}$ is countable.

23. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

   ___ If A is a countable set and B is an uncountable set, then $A \cup B$ is uncountable.

   ___ For any real numbers a, b such that $a < b$, the closed interval $[a, b]$ is uncountable.

   ___ If $A \subsetneq B$ then $|A| < |B|$.

   ___ The set $\{1, -2, 3, -4, 5, -6, \ldots\}$ is countable.

24. [2] An airline surveyed 200 passengers and recorded the following information about the type of non-alcoholic beverages they liked: 78 passengers liked coffee, 70 liked tea, and 66 liked orange juice; 35 liked both coffee and tea, 40 liked both tea and orange juice, 15 liked both coffee and orange juice, and 15 liked all three types of beverages. How many passengers liked only orange juice or none of the three beverages?

25. [2] Let $A = \{1, 2, 3, 4, 5\}$. Fill in each blank. No justification is necessary.

    (a) $|A \times A| =$ __________________.

    (b) The number of subsets of A that contain neither 1 nor 2 is __________________.

    (c) The number of functions $f : A \to \{w, x, y, z\}$ where $f(1) = w$, $f(2) = z$ and $f(5) \neq x$, is __________________.

    (d) The number of relations on A is __________________.

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