UNIVERSITY OF VICTORIA
APRIL EXAMINATIONS 2016

MATH 122: Logic and Foundations

CRN: 22116 (A01), 22117 (A02), 22118 (A03), and 24254 (A04)

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NAME: ______________________________

V00#: ______________________________

Duration: 3 Hours.

Answers are to be written on the exam paper.
No calculator is necessary, but a Sharp EL-510R or a Sharp EL-510RNB calculator is allowed.

This exam consists of 26 questions, for a total of 80 marks. Unless otherwise specified, in order to receive full or partial credit, you must show your work and justify your answers.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [4] Suppose the statement $p \leftrightarrow q$ is false. Find all combinations of truth values for p, q, and r such that the statement $(\neg r \leftrightarrow p) \vee (r \wedge \neg q)$ is true.

2. [3] Use known logical equivalences to show that $q \rightarrow (p \rightarrow q)$ is a tautology.

3. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

   ___ The negation of the statement $p \rightarrow q$ is $p \wedge \neg q$.

   ___ The converse of the statement $p \rightarrow \neg q$ is $p \rightarrow q$.

   ___ The statement $\exists x, (x^2 = 2) \rightarrow (2x < x)$ is true for the universe $\mathbb{Q}$.

   ___ The statement "*Every picture tells a story*" has a hidden existential existential quantifier.

4. [4] Use known logical equivalences and rules of inference to show that the following argument is valid.

$$\begin{array}{rc} & \neg(p \wedge q) \\ & r \vee q \\ & \neg p \to \neg r \\ \hline \therefore & p \leftrightarrow r \end{array}$$

5. [3] Let A, B and C be sets. Give a counterexample to show that the statement "*If* $A \cup C = B \cup C$ *then* $A = B$." is false.

6. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. The letters A and B represent sets.

___ If $x \in A \cap B$, then $(x, x) \in A \times B$.

___ $\mathcal{P}(\emptyset) = \emptyset$.

___ If $A \neq B$, then $A \cap B \subsetneq A \cup B$.

___ If the symmetric difference of A and B is not empty, then $A \not\subseteq B$.

7. Let A, B and C be sets. Suppose that $A \subseteq B$ and $B \subseteq C$.

   (a) [3] Prove that $A \subseteq C$.

   (b) [1] For the sets A, B and C as above, suppose that $(B \setminus A) \neq \emptyset$. Is it true that $A \subsetneq C$? Explain.

8. [4] Let A and B be sets. Use any method except a Venn diagram to prove the identity $(A \cap B)^c = A^c \cup B^c$. Hint: there is a short argument that uses set-builder notation.

9. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

   ___ For any set A, there exists a reflexive relation on A.

   ___ If $\mathcal{R}$ is a symmetric relation on a set A with at least two elements, then there exist $a, b \in A$ such that $(a, b), (b, a) \in \mathcal{R}$.

   ___ If $\mathcal{R}$ is a relation on $\{a, b, c\}$ such that $(a, b) \in \mathcal{R}$ and $(b, a) \notin \mathcal{R}$, then $\mathcal{R}$ is antisymmetric.

   ___ If $\mathcal{R}$ is a transitive relation on $\{a, b, c\}$ such that $(a, b) \in \mathcal{R}$ and $(a, c) \notin \mathcal{R}$, then $(b, c) \notin \mathcal{R}$.

10. [4] Let $\sim$ be the relation on $\mathbb{N}$ defined by $a \sim b$ if and only if $\frac{a}{b+2} \geq \frac{b}{a+2}$. Prove that $\sim$ is reflexive and anti-symmetric.

11. [2] Suppose that $\mathcal{R}$ is an equivalence relation on the set $A = \{a, b, c, d, e\}$ such that the partition of A into equivalence classes determined by $\mathcal{R}$ is $\{\{a, b, c\}, \{d\}, \{e\}\}$. Write $\mathcal{R}$ as a set of ordered pairs.

12. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

___ Suppose $a, b, q, r \in \mathbb{Z}$ and $b < 0$. If $a = bq + r,\ 0 \leq r < |b|$, then $q = \lceil a/b \rceil$.

___ If $|A| = 10$ and $f : A \to B$ is onto, then $|B| = 10$.

___ A function $f : A \to B$ is 1-1 if and only if for every $a \in A$ there exists exactly one $b \in B$ such that $(a, b) \in f$.

___ If the function $f : A \to B$ is invertible, then the inverse function, $f^{-1} : B \to A$ is 1-1 and onto.

13. [4] Let $f : \{0, 1, \ldots\} \to \{0, 1, \ldots\}$ be defined by $f(x) = x^2 - 5x + 6$. Show that f is neither 1-1 nor onto.

14. Let $X = \{a, b, c, d, e, f\}$, and let $f : X \to X$ be given in the table below

$x =$	a	b	c	d	e	f
$f(x) =$	b	c	a	f	d	e

(a) [1] Find $g = f \circ f$.

$x =$	a	b	c	d	e	f
$f(x) =$	b	c	a	f	d	e
$g(x) =$						

(b) [2] Use any method to show that $g = f^{-1}$.

15. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

___ When -17 is divided by 4, the remainder is -1.

___ If p and q are different prime numbers, then $lcm(p, q) = pq$.

___ $(1101101)_2 = (A5)_{16}$.

___ If $3k = 15\ell$, then $5 \mid k$.

16. Consider the integers $m = 900$ and $n = 189 = 3^3 7$. Fill in the blanks:

   (a) [1] The prime factorization of m is __________.

   (b) [1] $\gcd(m, n) =$__________ and $lcm(m, n) =$ __________.

   (c) [2] The smallest multiple k of 189 such that $\gcd(m, k) = 45$ is __________.

17. (a) [2] Prove that if $k \equiv 3 \pmod 8$, then $7 \cdot k^{2016} + 1$ is divisible by 8.

   (b) [2] Is there an integer $n > 1$ such that the last digit of 122^n is 2? Are the infinitely many such integers n?

18. [2] Use the blank to indicate whether each statement is True or False. All variables are integers. No justification is necessary.

   ___ $4 \mid (d_k d_{k-1} \ldots d_1 d_0)_{10}$ if and only if $4 \mid (d_1 d_0)_{10}$.

   ___ If $a \equiv 1 \pmod 6$ then $a \equiv 1 \pmod 3$.

   ___ If $p_1, p_2, \ldots, p_n$ are the first n prime numbers, then the integer $N = p_1 p_2 \cdots p_n + 1$ is prime.

   ___ It follows from $2016 = 7 \times 250 + 266$ that $\gcd(2016, 250) = \gcd(250, 266)$.

19. [4] Let a, b and c be integers such that $a + b + c = 0$. Let d be an integer such that $d \mid a$ and $d \mid b$. Prove that $d \mid c$.

20. [4] Use induction to prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$, for all integers $n \geq 1$.

21. [4] Let $a_0, a_1, \ldots$ be the sequence defined by $a_0 = 3$, and $a_n = 5a_{n-1} + 3$ for $n \geq 1$. Find a_1, a_2, a_3 and a_4, then use your work to conjecture (i.e. guess) a formula for a_n. It is not necessary prove that your formula is correct, but note that it is not sufficient to express a_n as a sum.

22. [4] Use the strong form of induction to prove that any integer $n \geq 8$ can be written as a sum of 3s and 5s. Note: the basis step should cover several values of n.

23. [3] Use a diagonal sweeping argument to prove that the set $\mathbb{N} \times \mathbb{N}$ is countable.

24. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

___ If A and B are infinite sets such that $|A| = |B|$, then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

___ If A and B are infinite sets such that $|A| = |B|$ and $a \in A$, then $|A \setminus \{a\}| < |B|$

___ The union of any two countable sets is countable.

___ $|\mathbb{Q}| = |(0, 1)|$.

25. (a) [1] Complete the statement of the Principle of Inclusion-Exclusion for two sets:

$$|A \cup B| = \underline{\hspace{4cm}}.$$

(b) [2] How many integers in the set $S = \{1, 2, 3, ..., 3000\}$ are divisible by 2 or 3?

26. [2] Let $A = \{1, 2, 3, 4, 5, 6\}$. Fill in each blank. No justification is necessary.

(a) $|A \times A| =$ __________________.

(b) The number of subsets $X \subseteq A$ such that $\{1, 2, 3\} \subseteq X$ is __________________.

(c) The number of functions $f : A \to \{w, x, y, z\}$ is __________________.

(d) The number of 1-1 functions $g : \{a, b, c\} \to A$ is __________________.

END