UNIVERSITY OF VICTORIA
APRIL EXAMINATIONS 2017

MATH 122: Logic and Foundations

CRN: 22073 [A01], 22074 [A02], 22075 [A03], 22076 [A04], 23980 [A05]

INSTRUCTORS: G. MacGillivray [A01], J. Manzer [A02, A05], J. Butterfield [A03], J. Huang [A04]

Please circle your instructor's name.

NAME: ______________________________

V00#: ______________________________

Duration: 3 Hours.

Answers are to be written on the exam paper.
No calculator is necessary, but a Sharp EL-510R or a Sharp EL-510RN calculator is allowed.

This exam consists of 27 questions, for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [3] Use a truth table to determine whether $\neg p \vee (q \rightarrow (p \wedge q))$ is a tautology. (Here, and elsewhere, be sure to clearly state your answer to the question!)

2. [4] Use known logical equivalences to show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$.

3. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

   ___ The converse of $p \rightarrow \neg q$ is $\neg p \rightarrow q$.

   ___ If $p \rightarrow q$ is false, the truth value of $(\neg p \vee \neg q) \rightarrow (p \leftrightarrow q)$ is false.

   ___ The statement $\exists x, \exists y, (2x = -3y) \wedge (xy < 0)$ is true if the universe of x, y is $\mathbb{Z}$.

   ___ The fact that there are infinitely many prime numbers can be proved by showing that for every natural number n there exists a prime number p with $p > n$.

4. [4] Write the following argument in symbolic form, and then use known logical equivalences and inference rules to show that it is valid. Please clearly indicate which letters correspond to which statements. Justify each step.

I am not wearing a pink tie or I am wearing a red shirt.
If it is not Saturday, then I am wearing a pink tie.
I am not wearing a red shirt

$\therefore$ It is Saturday

5. [3] Explain in detail why $n = 9$ is a counterexample to the following argument.

If n is prime, then n is not a sum of two odd numbers.
If n is even, then it is a sum of two odd numbers.
n is odd or n is a sum of two odd numbers.

$\therefore$ If n is not prime, then n is even.

6. [3] Prove that for all sets A, B, and C, $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$.

7. [4] Let A and B be sets such that $A \cup B = B$. Prove that $A \cap B = A$.

8. [2] Let $A = \{1, \{2, 3\}, \{1, 2, 3\}\}$ and $B = \{1, 2, \{1, 2, 3\}\}$. Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

___ $A \subseteq B$.

___ $\emptyset \subseteq A \setminus B$.

___ $2 \in A \cap B$.

___ $|A \cup B| = 6$.

9. [3] Give a counterexample to show the following statement is false: *For all sets A, B and C, $(A\backslash B) \cup (B\backslash C) = A\backslash C$.* You should clearly define your sets and explain why your example shows that the statement is false.

10. [2] Let $\mathcal{R}$ be the relation on $\mathbb{Z}$ defined by $(x, y) \in \mathcal{R} \Leftrightarrow 3x + 4y = 1$. Prove that $\mathcal{R}$ is antisymmetric.

11. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

    ___ Any relation on $\{1, 2, 3\}$ contains at most 2^3 ordered pairs.

    ___ If $\mathcal{R}$ is an antisymmetric relation on $\mathbb{Z}$ and $(1, 2) \notin \mathcal{R}$, then $(2, 1) \in \mathcal{R}$.

    ___ The relation $\mathcal{R} = \{(1, 1), (2, 2), (3, 3)\}$ on $\{1, 2, 3\}$ is transitive.

    ___ If $\sim$ is an equivalence relation and $x \sim y$, then the equivalence class of x equals the equivalence class of y.

12. (a) [3] Let $\sim$ be the relation on $\mathbb{Z}$ defined by $a \sim b \Leftrightarrow 6 \mid a^2 - b^2$. Prove that $\sim$ is an equivalence relation.

(b) [1] Which of $1, 2, \ldots, 7$ belong to the equivalence class $[11]$?

13. [2] Show that the function $f : \mathbb{N} \to \{4, 5, \ldots\}$ defined by $f(x) = x^2 + 3$ is not onto.

14. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

___ If $f : A \to B$ is onto then the range of f is B.

___ Every function $g : \{a, b, c, d, e\} \to \{x, y, z\}$ is onto.

___ There is a bijection $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ if and only if $m = n$.

___ If $f : A \to B$ is 1-1, then so is $\iota_B \circ f$.

15. [4] Suppose $f : A \to B$ and $g : B \to C$ are both 1-1 functions. Prove that $g \circ f : A \to C$ is 1-1.

16. [3] Let $A = \{1, 2, 3\}, B = \{w, x, y, z\}, C = \{r, s, t\}$. Let $f : A \to B$ be given by $f = \{(1, x), (3, z), (2, y)\}$. Find a function $g : B \to C$ such that $g \circ f : A \to C$ is $g \circ f = \{(1, r), (2, s), (3, t)\}$. Explain why $g \circ f$ is invertible and find $(g \circ f)^{-1}$.

17. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

___ If $a \in \mathbb{Z}$ and $5|a^3$ then $5|a^2$.

___ If $a, b \in \mathbb{N}$ then $gcd(a, b) \mid lcm(a, b)$.

___ If p is prime and $a \in \mathbb{Z}$ then $gcd(a, p) \in \{1, p\}$.

___ $5 \mid (2345)_{15}$.

18. [4] Prove that divisibility is a transitive relation on $\mathbb{Z}$. That is, prove that for all $a, b, c \in \mathbb{Z}$, if $a|b$ and $b|c$, then $a|c$.

19. [3] Find all ordered pairs of integers (c, d) such that $(cd)_5 = (dc)_9$.

20. [2] Use the blank to indicate whether each statement is True (T) or False (F). All variables are integers. No justification is necessary.

___ If p is an odd prime number, then $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$.

___ $(d_2d_1d_0)_7 \equiv d_2 + d_1 + d_0 \pmod 3$.

___ If $a \equiv 5 \pmod 6$, then $a^3 \equiv 2 \pmod 3$.

___ The integer n is odd if and only if $\lfloor n/2 \rfloor < \lceil n/2 \rceil$.

21. [4] Use the Euclidean Algorithm to show that $gcd(2017, 122) = 1$ and then use your work to find integers x and y such that $2017x + 122y = 1$.

22. [3] Suppose $k \equiv 3 \pmod 5$. Find the remainder when $11k^3 + 12$ is divided by 5.

23. [3] Let p be a prime number. Prove that if $p|ab$, then $p|a$ or $p|b$. Give an example to show that the statement can be false if p is not prime.

24. [4] Use induction to prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \geq 1$.

25. In a group of 348 students, 321 liked Math 122, and 286 liked Math 101. Sixteen students liked neither of these courses.

(a) [1] How many of the students liked both courses?

(b) [1] How many of the 348 students did not like Math 101?

26. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 2,\ b_1 = 5$ and $b_n = 5b_{n-1} - 6b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = 2^n + 3^n$ for all $n \geq 0$.

27. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

    ___ The number of proper subsets of a finite set A equals the number elements of the power set of A minus 1.

    ___ There is no set A for which there are exactly 2^{10} relations on A.

    ___ If $|A| = 7$ and $|B| = 10$, then $|A \times B| = 17$.

    ___ The number of 1-1 functions $f : \{1, 2, \ldots, 5\} \to \{1, 2, \ldots, 100\}$ is 100!.

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