UNIVERSITY OF VICTORIA
APRIL EXAMINATIONS 2024

MATH 122: Logic and Foundations

CRN: 22013 (A01), 22014 (A02), 22015 (A03), 22016 (A04), 24097 (A05)

Please circle your instructor and section:

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NAME: ______________________________

V00#: ______________________________

Duration: 3 Hours.

Answers are to be written on the exam paper.
No calculator is necessary, but a Sharp calculator with model identifier beginning EL-510R is allowed.

This exam consists of 25 questions, for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [3] Use any method to determine whether $p \to (q \to p)$ is a tautology.

2. [4] Use known logical equivalences to show that $\neg(\neg p \wedge \neg q) \wedge (p \vee \neg r)$ is logically equivalent to $p \wedge (r \to q)$.

3. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

    ___ If $\neg p \Leftrightarrow \neg q$, then $p \Rightarrow q$.

    ___ For the universe $\mathbb{Z}$, the negation of $\forall x, \exists y, (x \neq 0) \to (xy = 1)$ is $\exists x, \forall y, (x \neq 0) \wedge (xy \neq 1)$.

    ___ The contrapositive of "*If this question is confusing or technical, then it is hard.*" is "*If this question is not hard, then it is not confusing or not technical.*".

    ___ The conclusion of a valid argument is a tautology.

4. [4] Use known logical equivalences and inference rules to show that the following argument is valid.

$$\begin{array}{rl} & \neg q \vee p \\ & \neg r \to q \\ & \neg r \\ \hline \therefore & p \end{array}$$

5. [3] Give a counterexample to show that the following argument is invalid.

$$\begin{array}{rl} & p \to \neg q \\ & q \to \neg s \\ & \neg r \to s \\ & p \leftrightarrow r \\ \hline \therefore & \neg p \end{array}$$

6. [2] Let A, B and C be sets. Use the blank to indicate whether each statement is True or False. No justification is necessary.

___ If $A \subsetneq B$, then $B \neq \emptyset$.

___ If $A \subseteq \mathbb{Z}$ and $B \subseteq \mathbb{Q}$, then $A \cap B = \emptyset$.

___ If $A \neq \emptyset$ and $B \neq \emptyset$, then $A \times B \neq \emptyset$.

___ If $B, C \in \mathcal{P}(A)$ then $B \cup C \in \mathcal{P}(A)$.

7. (a) [3] Let A and B be sets such that $A \cup B \subseteq A \cap B$. Show that $A \subseteq B$ using an argument that starts with "Take any $x \in A \ldots$.

   (b) [1] Must the sets A and B in part (a) be equal? Explain why or why not.

8. [4] Let A and B be sets. Show that $(A \setminus B^c) \cup (A^c \cup B)^c = A$. Hint: set-theoretic identities.

9. [2] Let $A = \{1, 2, 3, 4, 5\}$. Fill in each blank. No justification is necessary.

   (a) The number of subsets of A that contain 4 and do not contain 5 is ____________.

   (b) The number of onto functions $f : A \to \{v, w, x, y, z\}$ where $f(1) = x$, $f(2) = w$, $f(3) = z$ and $f(5) \neq v$, is ____________.

   (c) $|A \times A| =$ ____________.

   (d) The number of relations on A is ____________.

10. [2] On a shelf of 50 books, 25 are non-fiction, 21 have a blue spine, and 26 are written by women. Of these, 10 are non-fiction and have a blue spine, 14 have a blue spine and are written by a woman, 11 are non-fiction books by women, and 6 are non-fiction with a blue spine written by women. How many books on the shelf fit into none of these three categories?

11. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. All variables are integers.

    ___ If $a \mid b$ and $b \mid c$, then $ab \mid c$.

    ___ If $a = 2^5 \cdot 3^2 \cdot 13^4$ and $b = 2^8 \cdot 11^3$, then $\operatorname{lcm}(a, b) = 2^8 \cdot 3^2 \cdot 11^3 \cdot 13^4$.

    ___ If $\gcd(a, b) = 3$ then there exist integers x and y such that $ax + by = 12$.

    ___ If p is a prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.

12. [4] Use the Euclidean Algorithm to find $d = \gcd(1442, 630)$ and then use your work to find integers x and y such that $1442x + 630y = d$.

13. [4] Find the base 7 representation of 2024.

14. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. All variables are integers.

___ The last digit of 7^{122} is 9.

___ If a is odd, then $a \equiv 1 \pmod 3$.

___ If $3a \equiv 3b \pmod{15}$, then $a \equiv b \pmod{15}$.

___ $(101011)_2 = (2B)_{16}$.

15. [4] Let $n = (d_2d_1d_0)_{10}$. Prove that $3 \mid n$ if and only if $3 \mid (d_2 + d_1 + d_0)$.

16. [4] Use induction to prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for all integers $n \geq 1$.

17. [4] Let $z_1, z_2, \ldots$ be the sequence defined by $z_1 = 1$, and $z_n = z_{n-1} + 2n$ for $n \geq 2$. Find z_2, z_3, z_4 and z_5, then use your work to obtain a formula for z_n (note: a formula, not just a summation). It is not necessary prove that your formula is correct.

18. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 2,\ b_1 = 9$ and $b_n = 9b_{n-1} - 20b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = 4^n + 5^n$ for all $n \geq 0$.

19. [4] Let $f : \mathbb{Z} \to \mathbb{N}$ be defined by $f(x) = x^2 + 2$. Explain why f is neither 1-1 nor onto.

20. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

    ___ For any $x \in \mathbb{R}$, $\lceil x \rceil - 1 = \lfloor x \rfloor$.

    ___ There exists an onto function from $\{1, 2, 3\}$ to $\mathbb{Z}$.

    ___ If $a = b$ implies $f(a) = f(b)$, then the function $f : A \to B$ is 1-1.

    ___ If A and B are finite, and $f : A \to B$ is 1-1 and onto, then $|A| = |B|$.

21. [4] For sets A, B and C, let $f : A \to B$ and $g : B \to C$ be onto functions. Prove that $g \circ f : A \to C$ is onto.

22. [5] Let $A = \{-10, -9, \ldots, -2, -1, 1, 2, \ldots, 9, 10\}$, and let $\sim$ be the relation on A defined by $a \sim b \iff ab > 0$. Prove that $\sim$ is an equivalence relation on A and find the partition of A it determines (i.e., determined by the different equivalence classes with respect to $\sim$).

23. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

    ___ The relation $\{(1,1),(1,2),(1,3),(1,4)\}$ on $\{1,2,3,4\}$ is reflexive.

    ___ For a transitive relation on $\mathbb{Z}$, if $(1,2) \in \mathcal{R}$ and $(2,3) \notin \mathcal{R}$, then $(1,3) \notin \mathcal{R}$.

    ___ The relation $\mathcal{R}$ on $\{2,4,6,\ldots\}$ defined by $(x,y) \in \mathcal{R}$ if and only if $x - y = 2$ is antisymmetric.

    ___ If $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$ such that $1 \mathcal{R} 3$ and $4 \mathcal{R} 3$, then $4 \mathcal{R} 1$.

24. [3] Let $A = \{1,3,5,7,9,\ldots\}$. Use a diagonal sweeping argument to prove that $A \times A$ is countable.

25. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

    ___ If A is an uncountable set and $A \subseteq B$, then B is an uncountable set.

    ___ For any real number $b > 0$, the open interval $(0,b)$ is uncountable.

    ___ The closed interval of real numbers $[-1,1]$ is uncountable..

    ___ The set $\{1, -\frac{1}{2}, 3, -\frac{1}{4}, 5, -\frac{1}{6}, 7, -\frac{1}{8}, \ldots\}$ is countable.

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