**UVic – MATH 122 – Logic and Foundations** **Final Exam**
**Date: Apr. 15, 2025** **Time Limit: 3 hours**

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Before starting your test, enter your name and student ID (clearly) on this page.

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| This exam contains 22 pages (including this one, and scrap pages) and 27 questions, worth a total of 80 marks. |

You may not open the exam until instructed to do so. In the meantime, please read the following instructions:

- No textbooks or class notes are allowed on this exam. A calculator is not necessary, but a Sharp EL-510R, EL-510RN, or EL-510RNB calculator is allowed.
- Answer each question in the space immediately below that question. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

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If you have work on this page you want marked,
please indicate this next to the original question.

[3] 1. Use any method to determine whether $(p \vee q) \to (p \to q)$ is a tautology. Write a sentence that explains your conclusion.

[3] 2. Use laws of logic and known logical equivalences to show that $(p \wedge \neg q) \to r$ is logically equivalent to $p \to (q \vee r)$.

[2] 3. Let $A$, $B$, and $C$ be sets. Check the appropriate circle to indicate whether each statement is true or false. No justification is necessary.

| | True | False | |
|---|---|---|---|
| (a) | ◯ | ◯ | If $p \to (q \vee r)$ is false, then $q$ is false. |
| (b) | ◯ | ◯ | If an implication is true, so is its converse. |
| (c) | ◯ | ◯ | If $p$ and $q$ are logically equivalent, then $\neg p \Rightarrow \neg q$. |
| (d) | ◯ | ◯ | The negation of the statement "All squares are rectangles." is the statement "There exists a rectangle that is not a square." |

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[2] 4. Suppose the universe is $\mathcal{U} = \{1, 2, 3, 4\}$. Determine the truth value of the statement $\exists\, x, \forall\, y, (y \geq 3) \to (x^2 < 7)$. Explain your reasoning.

[4] 5. Use known logical equivalences and inference rules to show that the following argument is valid.

$$\begin{array}{r} p \to \neg q \\ r \vee p \\ q \\ \hline \therefore r \end{array}$$

[3] 6. Give a counterexample to show that the following argument is invalid. Be sure to explain why what you have shown demonstrates that it is invalid.

$$\begin{array}{r} p \to q \\ r \to q \\ \hline \therefore r \to p \end{array}$$

[2] 7. Check the appropriate circle to indicate whether each statement is true or false. No justification is necessary. Let $A = \{\alpha, \beta, \{\gamma\}, \{\alpha, \gamma\}\}$.

| | **True** | **False** | |
|---|---|---|---|
| (a) | ○ | ○ | $\varnothing \subseteq A$. |
| (b) | ○ | ○ | $\gamma \in A$. |
| (c) | ○ | ○ | $\{\alpha, \beta\} \in \mathcal{P}(A)$. |
| (d) | ○ | ○ | $\{\alpha, \gamma\} \subseteq A$. |

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[4] 8. Let $A$ ane $B$ be sets such that $A \cup B = B$.

(a) Prove that $A \subseteq B$ using an argument that starts with "Let $x \in A$...".

(b) Use the universe $\mathcal{U} = \{x, y\}$ to give an example that shows that the sets $A$ and $B$ need not be equal.

[4] 9. Let $A$, $B$, and $C$ be sets. Using any method, prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.
*Hint:* There is a short proof using the laws of set theory.

[2] 10. Check the appropriate circle to indicate whether each statement is true or false. No justification is necessary. Let $A$, $B$, and $C$ be sets.

| | **True** | **False** | |
|---|---|---|---|
| (a) | ◯ | ◯ | If $A \subsetneq B$, then $B \not\subseteq A$. |
| (b) | ◯ | ◯ | If $A \cup B = A \cup C$, then $B = C$. |
| (c) | ◯ | ◯ | If $B \subseteq A \oplus B$, then $A = \varnothing$. |
| (d) | ◯ | ◯ | If $A \subseteq B$, then $A \times C \subseteq B \times C$. |

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[2] 11. Cetatea is an infraorder of acquatic mammals that includes whales, dolphins, and porpoises. Of the 95 species of cetaceans, 33 are under threat, and 6 of the species under threat can be found in the waters of British Columbia. There are 45 species which are neither under threat, nor can be found in BC. How many cetaceans can be found in British Columbia?

[3] 12. Let $a, b, c \in \mathbb{Z}$. Prove that if $a \mid b^2$ and $b^4 \mid c^3$, then $a^2 \mid c^3$.

[2] 13. Check the appropriate circle to indicate whether each statement is true or false. No justification is necessary. Let $a, b, c, d \in \mathbb{Z}$.

| | **True** | **False** | |
|---|---|---|---|
| (a) | ○ | ○ | Suppose $a + b = c$. If $d \mid c$ and $d \mid a$, then $d \mid b$. |
| (b) | ○ | ○ | $\gcd(a, b) \mid \operatorname{lcm}(a, b)$ |
| (c) | ○ | ○ | If $a \mid b^2$, then $a \mid b$. |
| (d) | ○ | ○ | Let $p$ be a prime. If $p \mid ab$, then $p \mid a$ or $p \mid b$. |

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[4] 14. Use the Euclidean Algorithm to find $d = \gcd(632, 288)$, and then use your work to find integers $x$ and $y$ such that $632x + 288y = d$.

[2] 15. Find the base 11 representation of 2025. If needed, use $\alpha$ to represent the digit 10.

[2] 16. Check the appropriate circle to indicate whether each statement is true or false. No justification is necessary. Let $a, b \in \mathbb{Z}$.

| | **True** | **False** | |
|---|---|---|---|
| (a) | ◯ | ◯ | If $\gcd(a, b) = 2$, then there exist $x, y \in \mathbb{Z}$ such that $ax + by = 10$. |
| (b) | ◯ | ◯ | If $a \equiv 7 \pmod{12}$, then $a \equiv 1 \pmod{3}$. |
| (c) | ◯ | ◯ | If $b$ is even, then $b^2 \equiv 0 \pmod{4}$. |
| (d) | ◯ | ◯ | Let $n \in \mathbb{N}$ and $r \in \mathbb{R}$ with $r > 1$. Then $1+r+r^2+\cdots+r^n = \dfrac{r^{n+1}-1}{r-1}$. |

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[3] 17. Find the last digit of $3^{250}$ in base 5, that is if $3^{250} = (d_n d_{n-1} \ldots d_0)_5$, find $d_0$.

[4] 18. Use induction to prove that $3^n \geq n^2$ for all $n \geq 2$.

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[4] 19. Let $z_n$ denote the recursively defined sequence where $z_1 = 0$, and $z_{n+1} = 4z_n + 2$ for all $n \geq 1$.

(a) Find $z_2$, $z_3$, $z_4$, and $z_5$. You may leave your answers unsimplified.

(b) Use your work in (a) to find a formula for $z_n$ that depends only on $n$, and is valid for all $n \geq 1$. Your answer should not involve a sum of many terms. You do not need to prove that your formula is correct.

[4] 20. Let $a_n$ denote a sequence recursively defined by $a_1 = 1$, $a_2 = 8$ and $a_n = 6a_{n-1} + 16a_{n-2}$ for all $n \geq 3$. Prove that $a_n = 8^{n-1}$ for all $n \geq 1$.

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[4] 21. Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = x^2 + 1$. Prove that $f$ is $1 - 1$ (injective) but not onto (surjective). Carefully explain your reasoning. Note that $[0, \infty) = \{x : x \geq 0, x \in \mathbb{R}\}$.

[2] 22. Check the appropriate circle to indicate whether each statement is true or false. No justification is necessary.

| | True | False | |
|---|---|---|---|
| (a) | ◯ | ◯ | A function $f \subseteq A \times A$ is never an equivalence relation. |
| (b) | ◯ | ◯ | There are exactly $2^4$ possible relations that can be defined on a set of 2 elements. |
| (c) | ◯ | ◯ | The relation $\mathcal{Q}$ on $\mathbb{Z}$ defined such that $(a, b) \in \mathcal{Q}$ if and only if $a \equiv b \pmod 7$ partitions $\mathbb{Z}$ into 6 equivalence classes. |
| (d) | ◯ | ◯ | There are no relations on any given set that are simultaneously symmetric and antisymmetric. |

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[4] 23. For sets $A, B$ and $C$, let $f : A \to B$ and $g : B \to C$ be onto (surjective) functions. Prove that $g \circ f : A \to C$ is onto.

[4] 24. Define the relation $\mathcal{R}$ on $\mathbb{R}$ such that $(x, y) \in \mathcal{R}$ if and only if $x - y \in \mathbb{Z}$. Prove that $\mathcal{R}$ is an equivalence relation.

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[2] 25. Consider the set $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$ with the relation $\mathcal{R}$ on $A$ defined by $(a, b) \in \mathcal{R}$ if and only if $a^2 - b^2$ is divisible by 3. The relation $\mathcal{R}$ is an equivalence relation (you do not need to show this). Find all the equivalence classes of $\mathcal{R}$.

[3] 26. Prove that $\{n^{1/m} : n \in \mathbb{N}, m \in \mathbb{N}\}$ is countable using a diagonal sweeping argument. *Hint:* In the proof that $\mathbb{N} \times \mathbb{N}$ is countable, the ordered pair $(n, m)$ appeared in row $n$ and column $m$ of the array.

[2] 27. Check the appropriate circle to indicate whether each statement is true or false. No justification is necessary.

| | **True** | **False** | |
|---|---|---|---|
| (a) | ◯ | ◯ | Let $A$ be a set with 2 elements and let $B$ be a set with 3 elements. There are exactly two 1-1 (injective) functions $A \to B$. |
| (b) | ◯ | ◯ | A function $f$ has an inverse if and only if the relation $\{(b, a) : (a, b) \in f\}$ is a function. |
| (c) | ◯ | ◯ | The set $\{\sqrt{\lvert x \rvert} : x \in \mathbb{Q}\}$ is uncountable. |
| (d) | ◯ | ◯ | The set of odd integers is countable. |

[END OF EXAM]

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