

1. [3] Use a truth table to determine whether $(p \leftrightarrow \neg q) \rightarrow (\neg p \vee \neg q)$ is a tautology. (Here, and elsewhere, be sure to clearly state your answer to the question!)

p	q	$\neg p$	$\neg q$	$p \leftrightarrow \neg q$ ⁽¹⁾	$\neg p \vee \neg q$ ⁽²⁾	(1) \rightarrow (2)
T	F	F	T	F	T	T
T	T	F	F	T	T	T
F	F	T	T	T	T	T
F	T	T	F	F	F	T

\therefore Stmt is a tautology

all true

2. [4] Use known logical equivalences to show that $\neg[p \vee (\neg p \wedge q)]$ is logically equivalent to $\neg[p \vee q]$.

$$\begin{aligned}
 & \neg[p \vee (\neg p \wedge q)] \\
 \Leftrightarrow & \neg p \wedge \neg(\neg p \wedge q) && \text{De Morgan} \\
 \Leftrightarrow & \neg p \wedge (p \vee \neg q) && \text{De Morgan} \\
 \Leftrightarrow & (\cancel{\neg p \wedge p}) \vee (\neg p \wedge \neg q) && \text{Dist.} \\
 \Leftrightarrow & 0 \vee (\neg p \wedge \neg q) && \text{Known contra.} \\
 \Leftrightarrow & \neg p \wedge \neg q && \text{Identity} \\
 \Leftrightarrow & \neg(p \vee q) && \text{De Morgan}
 \end{aligned}$$

3. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

- The contrapositive of $p \rightarrow \neg q$ is $\neg p \rightarrow q$.
- If the statement $(p \vee q) \wedge (\neg r \vee q)$ is true, then q must be true.
- If $p \rightarrow q$ is true, then its converse is false.
- For the universe of integers, the statements $\forall x, \exists y, x = -y$ and $\exists y, \forall x, x = -y$ are both true.

4. [4] Use known logical equivalences and rules of inference to show the given argument is valid. Justify each step.

<p>1. $q \rightarrow p$</p> <p>2. $\neg(r \wedge p)$</p> <p>3. r</p> <p>4. $\neg p \rightarrow \neg q$</p> <p>5. $\neg r \vee \neg p$</p> <p>6. $r \rightarrow \neg p$</p> <p>7. $r \rightarrow \neg q$</p> <p>8. $\therefore \neg q$</p>	$\frac{q \rightarrow p \quad \neg(r \wedge p)}{r} \quad \therefore \neg q$ <p>Premise</p> <p>"</p> <p>"</p> <p>1, Contrapos.</p> <p>2, DeMorgan</p> <p>5, Known LE.</p> <p>6, 4, Chain rule.</p> <p>3, 7, M.P.</p>
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5. [3] Give a counterexample to show the following argument is invalid.

$\frac{p \rightarrow r \quad p \vee \neg q}{\therefore \neg r}$	<p>1</p> <p>1</p> <p>0</p>
<p>$\begin{pmatrix} p & q & r \\ 1 & 0 & 1 \end{pmatrix}$</p> <p style="margin-left: 40px;">↑ 0 or 1</p>	<p>The given Truth Assmt(s) result in all premises being T & the concl. being F \therefore arg not valid.</p>

6. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

- T If the integer n^2 is a multiple of 3, then n is a multiple of 3.
- T If $bk_1 + r_1 = bk_2 + r_2$, where $0 \leq r_1 < |b|$ and $0 \leq r_2 < |b|$, then $k_1 = k_2$ and $r_1 = r_2$. (assume these are integers)
- F The Fundamental Theorem of Arithmetic states that there are infinitely many prime numbers.
- T If the integer $n > 1$ has no prime divisor which is less than or equal to \sqrt{n} , then n is prime.

7. [4] Let a, b, c be integers. Prove that if $c \mid a$ and $c \mid (a + b)$, then $c \mid b$.

If c divides a & also divides $a+b$,
it divides $(a+b) - a = b$.

OR

Given $c \mid a$ and $c \mid a+b$.

$$\therefore \exists k_1 \in \mathbb{Z} \text{ s.t. } ck_1 = a$$

$$\text{ \& } \exists k_2 \in \mathbb{Z} \text{ s.t. } ck_2 = a+b$$

$$\therefore b = (a+b) - a = ck_2 - ck_1 \\ = c(k_2 - k_1)$$

Since $k_2 - k_1 \in \mathbb{Z}$, $c \mid b$.

8. [3] Find the base 12 representation of 1596. Use T and E to represent the digits 10 and 11, if necessary.

$$\begin{aligned} 1596 &= 12 \times 133 + 0 \\ 133 &= 12 \times 11 + 1 \\ 11 &= 12 \times 0 + \textcircled{11} E \end{aligned} \quad \uparrow$$

$$1596 = (E10)_{12}$$

9. [2] Use the blank to indicate whether each statement is True (T) or False (F). All variables are integers. No justification is necessary.

F For all $a, b \in \mathbb{Z}$, if $\gcd(a, b) = 1$ then $\text{lcm}(a, b) = ab$.

T $321 \equiv -123 \pmod{12}$.

T Let $d_k d_{k-1} \dots d_0$ be the base 10 representation of n .
Then $n \equiv d_k + d_{k-1} + \dots + d_0 \pmod{3}$

F The last digit in the base 10 representation of 122^{122} is 8.

$$\begin{array}{r} 26 \\ 12 \overline{) 321} \\ \underline{24} \\ 81 \\ \underline{72} \\ 9 \end{array}$$

10. [4] Use the Euclidean Algorithm to find $d = \gcd(630, 196)$ and then use your work to find integers x and y such that $630x + 196y = d$.

$$630 = 3 \times 196 + 42$$

$$196 = 4 \times 42 + 28$$

$$42 = 1 \times 28 + 14 \leftarrow \therefore \gcd(630, 196) = 14$$

$$28 = 2 \times 14 + 0$$

$$14 = 42 - 28$$

$$= 42 - (196 - 4 \times 42)$$

$$= 5 \times 42 - 196$$

$$= 5(630 - 3 \times 196) - 196$$

$$= 5 \times 630 - 16 \times 196$$

$$= 630(5) + 196(-16)$$

$x \nearrow$

$\nwarrow y$

11. [3] Let $a = 6!$ and $b = 10!$. Find the prime decomposition of a , and of b , and then use these to find $\text{lcm}(a, b)$.

$$a = 2 \times 3 \times 2^2 \times 5 \times (2 \cdot 3) = 2^4 3^2 5^1$$

$$b = 2 \times 3 \times 2^2 \times 5 \times (2 \cdot 3) \times 7 \times 2^3 \times 3^2 \times (2 \cdot 5) \\ = 2^8 3^4 5^2 7^1$$

$$\text{lcm}(2^4 3^2 5^1 7^0, 2^8 3^4 5^2 7^1) = 2^8 3^4 5^2 7^1$$

12. [3] Let p be a prime number, and $a, b \in \mathbb{Z}$. Suppose that $p|ab$, but p does not divide b . Prove, or explain why, $p|a$. Give an example to show that the statement can be false if p is not prime.

If $p|ab$, then p appears in the prime factorization of ab . We're given that p is not a prime factor of b (as $p \nmid b$)
 $\therefore p$ is prime factor of a
 $\therefore p|a$

Example $4|6 \cdot 10$ but $4 \nmid 6$

and $4 \nmid 10$

\therefore stmt can be false if p not prime

13. [4] Use induction to prove that $(n+1)! > 3^n$ for all integers $n \geq 4$.

Basis When $n=4$, LHS = $(4+1)! = 5! = 120$
 \neq RHS = $3^4 = 81$

LHS > RHS \therefore stmt true when $n=4$.

IH Suppose there is an integer $k \geq 4$
 s.t. $(n+1)! > 3^n$ for $n = 4, 5, \dots, k$.

IS. Want $((k+1)+1)! > 3^{k+1}$

Look at LHS:

$$\begin{aligned} (k+2)! &= (k+2)(k+1)! && \text{since } k \geq 4. \\ &> (k+2)3^k && \text{by IH} \\ &> 3 \cdot 3^k && \text{b/c } k+2 \geq 6 > 3 \\ &= 3^{k+1}, \text{ as wanted.} \end{aligned}$$

\therefore By PMI, $(n+1)! > 3^n \forall n \geq 4$

14. Let a_0, a_1, \dots be the sequence recursively defined by $a_0 = 2$, and $a_n = 7a_{n-1} + 2$ for $n \geq 1$.

- (a) [2] Find a_1, a_2, a_3 , and a_4 . Leave each of these as a sum rather than computing a numerical value.

$$\begin{aligned} a_0 &= 2 \\ a_1 &= 7a_0 + 2 = 7 \cdot 2 + 2 \\ a_2 &= 7a_1 + 2 = 7(7 \cdot 2 + 2) + 2 = 7^2 \cdot 2 + 7 \cdot 2 + 2 \\ a_3 &= 7a_2 + 2 = 7(7^2 \cdot 2 + 7 \cdot 2 + 2) = 7^3 \cdot 2 + 7^2 \cdot 2 + 7 \cdot 2 \\ a_4 &= 7a_3 + 2 = 7(7^3 \cdot 2 + 7^2 \cdot 2 + 7 \cdot 2 + 2) + 2 \end{aligned}$$

- (b) [2] Based on your work in part (a), guess a formula for a_n , for all integers $n \geq 0$.
 Give your final answer as a closed-form formula that is not a summation.

Guess

$$\begin{aligned} a_n &= 7^n \cdot 2 + 7^{n-1} \cdot 2 + \dots + 7^1 \cdot 2 + 2 \\ &= 2(7^n + 7^{n-1} + \dots + 7 + 1) \\ &= 2 \left(\frac{7^{n+1} - 1}{7 - 1} \right) = \frac{7^{n+1} - 1}{3} \end{aligned}$$

$$= 7^4 \cdot 2 + 7^3 \cdot 2 + 7^2 \cdot 2 + 7 \cdot 2 + 2$$

15. [4] Let a_1, a_2, \dots be the sequence recursively defined by $a_1 = 3, a_2 = 7$, and $a_n = 5a_{n-1} - 6a_{n-2}$, for $n \geq 3$. Use (the strong form of) induction to prove that $a_n = 2^n + 3^{n-1}$ for all $n \geq 1$.

Basis: When $n=1$, $a_1 = 3 \stackrel{!}{=} 2^1 + 3^{1-1} = 3 \checkmark$
 When $n=2$, $a_2 = 7 \stackrel{!}{=} 2^2 + 3^{2-1} = 7 \checkmark$
 \therefore Stmt true when $n=1$ & when $n=2$

IH Suppose there is an integer $k \geq 2$ s.t. $a_n = 2^n + 3^{n-1}$ for $n = 1, 2, \dots, k$.

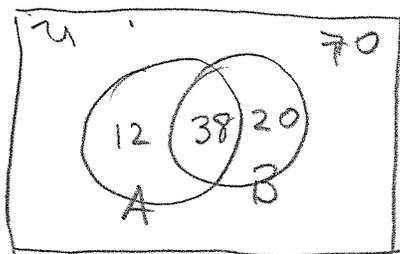
IS Want $a_{k+1} = 2^{k+1} + 3^{(k+1)-1} = 2^{k+1} + 3^k$.

Look at LHS

$$\begin{aligned} a_{k+1} &= 5a_k - 6a_{k-1} \quad \text{b/c } k+1 \geq 3 \\ &= 5(2^k + 3^{k-1}) - 6(2^{k-1} + 3^{k-2}) \quad \text{by IH} \\ &= 5 \cdot 2^k - 6 \cdot 2^{k-1} + 5 \cdot 3^{k-1} - 6 \cdot 3^{k-2} \\ &= 5 \cdot 2^k - 3 \cdot 2 \cdot 2^{k-1} + 5 \cdot 3^{k-1} - 2 \cdot 3 \cdot 3^{k-2} \\ &= 5 \cdot 2^k - 3 \cdot 2^k + 5 \cdot 3^{k-1} - 2 \cdot 3^{k-1} \\ &= 2 \cdot 2^k + 3 \cdot 3^{k-1} = 2^{k+1} + 3^k \quad \checkmark \end{aligned}$$

\therefore By PMI $a_n = 2^n + 3^{n-1} \quad \forall n \geq 1$

16. [3] Two sets A and B have elements in a universe with $|U| = 140$. If $|A \cup B| = 70$, $|A| = 50$, and $|B^c| = 82$. How many elements are in exactly one of the two sets?



$$B \cup B^c = U, \quad |B^c| = 82$$

$$\therefore |B| = 140 - 82 = 58$$

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ 70 &= 50 + 58 - |A \cap B| \end{aligned}$$

$$\therefore |A \cap B| = 38$$

From the diagram

$$|A \setminus B| + |B \setminus A| = 12 + 20 = 32$$

17. [4] Let A, B, C be sets. Prove that if $A \subseteq B$, then $A \times C \subseteq B \times C$.

Given: $A \subseteq B$ Want $A \times C \subseteq B \times C$

Take any $(x, y) \in A \times C$

$\therefore x \in A$ and $y \in C$

Since $A \subseteq B$, $x \in B$.

$\therefore (x, y) \in B \times C$

$\therefore A \times C \subseteq B \times C$

18. [3] Give a counterexample to show the following statement is false: "For any sets A, B, C , if $A \cap B = A \cap C$ then $B = C$."

If $A = \emptyset$, $B = \{1\}$, $C = \{2\}$
 then $A \cap B = \emptyset = A \cap C$ but $B \neq C$



19. [2] Let $A = \{1, \{2, 3\}, \{1, 3\}\}$ and $B = \{1, 2, 3, \{1, 2, 3\}\}$. Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

F $A \subseteq B$.

T $\emptyset \subseteq B$.

F $2 \in A$.

T $|A \cup B| = 6$.

$A \cup B = \{1, 2, 3, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$

20. Let \mathcal{R} be the relation on \mathbb{Z} defined by $(x, y) \in \mathcal{R} \Leftrightarrow x + y$ is even.

(a) [3] Prove that \mathcal{R} is an equivalence relation.

Reflexive: Take any $x \in \mathbb{Z}$.
Then $x+x = 2x$ is even
 $\therefore (x, x) \in \mathcal{R} \quad \therefore \mathcal{R}$ is refl.

Symmetric: Suppose $(x, y) \in \mathcal{R}$.
 $\therefore x+y$ is even $\therefore y+x$ is even
 $\therefore (y, x) \in \mathcal{R} \quad \therefore \mathcal{R}$ is Symm.

Transitive: Suppose $(x, y), (y, z) \in \mathcal{R}$.
 $\therefore x+y$ is even & $y+z$ is even
 $\therefore (x+y) + (y+z) = x + 2y + z$ is even
 $\therefore x + 2y + z = 2t$ for some $t \in \mathbb{Z}$.
 $\therefore x + z = 2t - 2y$, which is even
 $\therefore (x, z) \in \mathcal{R} \quad \therefore \mathcal{R}$ is trans.

(b) [1] Find, or describe, the equivalence class [122].

[122]
= $\{x \in \mathbb{Z}, x + 122 \text{ is even}\}$
= set of all even integers

$\therefore \mathcal{R}$ is an equiv. rel'n

21. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

There are $2^{(2^4)}$ relations on $\{1, 2, 3, 4\}$

The relation $\mathcal{R} = \{(1, 1), (1, 2), (3, 1), (3, 2)\}$ on $\{1, 2, 3\}$ is anti-symmetric.

For any non-empty set A , a function $f : \{1, 2, 3\} \rightarrow A$ contains exactly 3 ordered pairs.

There is no 1-1 function from \mathbb{Z} to $\{1, 2, \dots\}$.

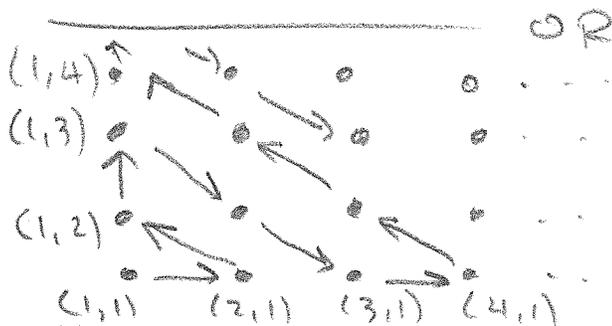
25. Consider the set $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{1, 2, \dots\}$.

(a) [1] Let $k \geq 2$ be an integer. How many ordered pairs $(a, b) \in \mathbb{N} \times \mathbb{N}$ are such that $a + b = k$?

$(1, k-1), (2, k-2), \dots, (k-1, 1)$
 $\therefore k-1$ of them

(b) [1] Using part (a) or some other method, describe a sequence that contains every element of $\mathbb{N} \times \mathbb{N}$ exactly once.

List the ordered pairs with sum 1, then with sum 2, and so on.



OR
 The array contains every element of $\mathbb{N} \times \mathbb{N}$ & the seq shown contains every element of the array

(c) [2] Explain why part (b) proves that $\mathbb{N} \times \mathbb{N}$ is a countably infinite set.

It is infinite & there is a sequence that contains all of its elements
 \therefore it is countably infinite.

26. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

- F The set of prime numbers does not have the same cardinality as the set of rational numbers.
- F The open interval $(0, 1)$ does not have the same cardinality as \mathbb{R} .
- F If X is an uncountable set and $Y \subseteq X$, then Y is uncountable.
- T If A, B, C are countable sets, then there exists a sequence containing every element of $A \cup B \cup C$ exactly once.

$A = \{a_1, a_2, \dots\}$
 $B = \{b_1, b_2, \dots\}$
 $C = \{c_1, c_2, \dots\}$
 The elements of $A \cup B \cup C$ can be listed $a_1, b_1, c_1, a_2, b_2, c_2, \dots$ /END

22. [3] Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $f(n) = \lceil n/3 \rceil$. Prove that f is onto.

Take any $y \in \mathbb{Z}$.

$$\text{Then } f(n) = y \iff \lceil \frac{n}{3} \rceil = y$$

If $n = 3y$

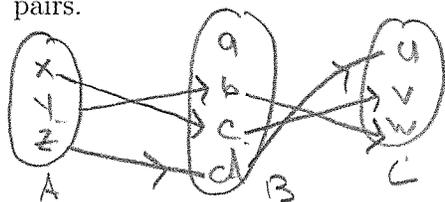
then $n \in \mathbb{Z}$, and

$$f(n) = f(3y) = \lceil \frac{3y}{3} \rceil = y$$

$\therefore f$ is onto

23. Let $A = \{x, y, z\}$, $B = \{a, b, c, d\}$, and $C = \{u, v, w\}$. Let $f : A \rightarrow B$ and $h : A \rightarrow C$ be $f = \{(x, c), (y, b), (z, d)\}$, $h = \{(x, v), (y, w), (z, u)\}$.

(a) [1] Find a function $g : B \rightarrow C$ such that $g \circ f = h$. Write g as a set of ordered pairs.



$$g = \{(a, u), (b, w), (c, v), (d, u)\}$$

u or v
or w

(b) [1] Write $f \circ h^{-1}$ as a set of ordered pairs.

$$h^{-1} = \{(v, x), (w, y), (u, z)\}$$

$$f \circ h^{-1} = \{(v, c), (w, b), (u, d)\}$$

24. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

The function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that $f(x) = 2x$ has an inverse.

The identity function $\iota : A \rightarrow A$ is an equivalence relation on A .

If there exists a 1-1 and onto function $f : A \rightarrow B$, then $|A| = |B|$.

If $a, b \in \mathbb{R}$ and $a < b$, then there is a bijection (i.e. a 1-1 correspondence) from $(0, 1)$ to (a, b) .