UNIVERSITY OF VICTORIA
DECEMBER EXAMINATIONS 2016

MATH 122: Logic and Foundations

CRN: 12190 [A01], 12191 [A02], 12192 [A03], 14105 [A04]

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NAME: Solutions (if jet lagged)

V00#: ______________________

Duration: 3 Hours.

Answers are to be written on the exam paper.
No calculator is necessary, but a Sharp EL-510R or a Sharp EL-510RNB calculator is allowed.

This exam consists of 27 questions, for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [3] Use a truth table to determine whether $(p \to \neg q) \leftrightarrow \neg(p \vee \neg q)$ is a contradiction.

p	q	$\neg q$	$\overset{S_1}{p \to \neg q}$	$\overset{S_2}{p \vee \neg q}$	$\neg S_2$	$S_1 \leftrightarrow \neg S_2$
F	F	T	T	T	F	F
F	T	F	T	F	T	T ←
T	F	T	T	T	F	
T	T	F	F	T	F	

not a contradiction b/c not always F

2. [4] Use known logical equivalences to show that $p \vee \neg(q \to \neg r)$ is logically equivalent to $(\neg q \to p) \wedge (\neg r \to p)$.

$(\neg q \to p) \wedge (\neg r \to p)$

$\Leftrightarrow (\neg\neg q \vee p) \wedge (\neg\neg r \vee p)$ Known LE.

$\Leftrightarrow (q \vee p) \wedge (r \vee p)$ Dbl neg'n x 2

$\Leftrightarrow (q \wedge r) \vee p$ Dist've.

$\Leftrightarrow p \vee \neg(\neg q \vee \neg r)$ Comm + De Morgan

$\Leftrightarrow p \vee \neg(q \to \neg r)$ Known LE.

3. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

F The contrapositive of the statement "*Its a long way to the top if you want to rock and roll*" is "*If you don't want to rock and roll, it isn't a long way to the top*".

T The assertion "*every logical statement has a dual*" contains a hidden existential quantifier.

F The statements $\exists y, \forall x, x + y = 0$ and $\forall x, \exists y, x + y = 0$ are logically equivalent for the universe $\mathbb{Z}$.

T The statement $\forall x, (x^2 \leq 0) \leftrightarrow (2x + 3 = 3)$ is true for the universe $\mathbb{R}$.

4. [4] Use known logical equivalences and inference rules to show that the following argument is valid.

$$\begin{array}{rc} & \neg(q \wedge p) \\ & \neg r \rightarrow p \\ & q \\ \hline \therefore & r \end{array}$$

1. $\neg(q \wedge p)$ Premise
2. $\neg r \rightarrow p$ "
3. q "
4. $\neg q \vee \neg p$ 1, DeMorgan
5. $q \rightarrow \neg p$ 4, Known LE
6. $\therefore \neg p$ 3, 5 M.P.
7. $\neg p \rightarrow r$ 2, Contrapositive
8. $\therefore r$ 6, 7, M.P.

5. [3] Write the following argument in symbolic form, and then give a counterexample to show that it is invalid. Remember to define the letters you use for statements!

Either I wear a red tie or I wear blue socks
I am wearing blue socks
I am not wearing a red tie

Let r: I wear a red tie
b: " " blue socks

Then the arg is

$$\begin{array}{rl} r \vee b & T \\ b & T \\ \hline \therefore \neg r & F \end{array}$$

$\begin{pmatrix} r & b \\ T & T \end{pmatrix}$

This T.A. makes all premises T & the concl. F
∴ arg. not valid

6. [2] Let A, B, and C be sets. Use the blank to indicate whether each statement is True or False. No justification is necessary.

T There are exactly three different sets X such that $\{1,3\} \subseteq X \subsetneq \{1,2,3,4\}$.

F If some element of A is an element of B, then $A \subseteq B$.

F If $A \subseteq C$ and $B \subseteq C$ then $A \nsubseteq B$.

F If $A \times B = A \times C$, then $B = C$.

7. [4] Suppose that $A \cup B = B$. Prove that $B^c \subseteq A^c$, using an argument that starts with "*Take any* $x \in B^c$ ...".

Suppose $A \cup B = B$.
Take any $x \in B^c$.
$\therefore\ x \notin B$
$\therefore\ x \notin A \cup B$ since $A \cup B = B$ & $x \notin B$
$\therefore\ x \notin A$
$\therefore\ x \in A^c$
$\therefore\ B^c \subseteq A^c$

8. [4] Let A, B and C be sets. Prove that $A \setminus (B \setminus C) = (A \setminus B) \cup (A \setminus C^c)$.

$A \setminus (B \setminus C)$
$= A \setminus (B \cap C^c)$ Known equality
$= A \cap (B \cap C^c)^c$ " "
$= A \cap (B^c \cup C)$ DeMurgan & dbl compl.
$= (A \cap B^c) \cup (A \cap C)$ Dist.
$= (A \setminus B) \cup (A \setminus C^c)$ Known equality

9. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

F Any relation on $\{1, 2, 3\}$ that contains at least 3 ordered pairs is reflexive.
T The relation $\mathcal{R} = \{(1,1), (2,2)\}$ on $\mathbb{N}$ is symmetric, antisymmetric and transitive.
T If $\mathcal{R}$ is relation on a set $\{1, 2, 3, 4, 5\}$ that is both symmetric and antisymmetric, then $\mathcal{R}$ contains at most 5 ordered pairs.
T If $\mathcal{R}$ and $\mathcal{S}$ are transitive relations on a set A, then $\mathcal{R} \cap \mathcal{S}$ is a transitive relation on A.

10. [2] Give an example of antisymmetric relations $\mathcal{R}$ and $\mathcal{S}$ on $A = \{1, 2, 3\}$ such that $\mathcal{R} \cup \mathcal{S}$ is not antisysmmetric.

Let $R = \{(1,2)\}$ and $S = \{(2,1)\}$.
Then R & S are antisymmetric
But $R \cup S = \{(1,2), (2,1)\}$ is not anti-symm b/c $1 \neq 2$.

11. [4] Let $\sim$ be the relation on the set of all subsets of $\{1, 2, 3, 4\}$ by $X \sim Y \Leftrightarrow$ the smallest element of X equals the smallest element of Y. Prove that $\sim$ is an equivalence relation and find the equivalence class $[\{2, 4\}]$.

refl. For any $X \subseteq \{1,2,3,4\}$, the smallest element of X equals the smallest element of X (even if $X = \phi$) $\therefore \sim$ is reflexive.

Symm. Suppose $X \sim Y$.
$\therefore$ The smallest element of X equals the smallest element of Y
$\therefore$ The smallest el. of Y equals the smallest el. of X
$\therefore Y \sim X$ $\therefore \sim$ is symmetric.

trans Suppose $X \sim Y$ & $Y \sim Z$
$\therefore$ The smallest el of X equals the smallest el of Y, & the smallest el. of Y equals the smallest el of Z
$\therefore$ the smallest el. of X equals the smallest el. of X
$\therefore X \sim Z$ $\therefore \sim$ is transitive

$\therefore \sim$ is an equivalence rel'n

$[\{2,4\}] = \{\{2,3\}, \{2,4\}, \{2,3,4\}, \{2\}\}$

12. Let A and B be sets. Complete the following definitions.

(a) [1] A function $f : A \to B$ is: a subset of $A \times B$ s.t. every element of A is the 1^{st} comp. of exactly 1 ordered pair in f.

(b) [1] A function $f : A \to B$ is 1-1 if: $f(a) = f(b)$ implies $a = b$.

(c) [1] A function $f : A \to B$ is onto if: for every $b \in B$ there exists $a \in A$ s.t. $f(a) = b$.

13. (a) [4] Suppose a and b are real numbers such that $a < b$. Let $f : (0, 1) \to (a, b)$ be defined by $f(x) = (b - a)x + a$. Prove that f is invertible.

To show: f is 1-1 & onto.

1-1. Suppose $f(x_1) = f(x_2)$

$\therefore (b-a)x_1 + a = (b-a)x_2 + a$

$\therefore (b-a)x_1 = (b-a)x_2$

$\therefore x_1 = x_2$ b/c $b - a \neq 0$.

$\therefore f$ is 1-1

onto Take any $y \in (a,b)$.

If $f(x) = y$ then $(b-a)x + a = y$.

$\therefore (b-a)x = y - a$

$\therefore x = \dfrac{y-a}{b-a}$

$y \in (a,b) \therefore y > a \therefore y - a > 0 \therefore \dfrac{y-a}{b-a} > 0$

also $y < b \therefore y - a < b - a \therefore \dfrac{y-a}{b-a} < 1$

$\therefore x \in (0,1)$

$\therefore f$ is onto

(b) [1] Find a formula for f^{-1}.

From our work in (a)

$f^{-1}(y) = \dfrac{y-a}{b-a} = x$.

14. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

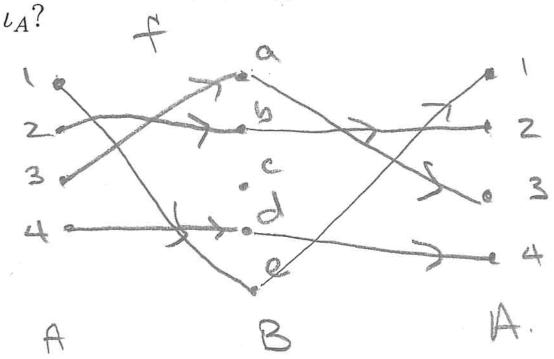
F If $f : A \to B$ is a function, then B is the range of f.

F $f \circ g = g \circ f$ for all functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$.

T There exists a 1-1 function $f : \mathbb{Z} \to \mathbb{N}$.

T If $f : A \to B$ and $g : B \to C$ are both onto, then $g \circ f$ is onto.

15. [2] Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$. Suppose that $f : A \to B$ is $f = \{(1, e), (2, b), (3, a), (4, d)\}$. How many different functions $g : B \to A$ are such that $g \circ f = \iota_A$?



Need $(e,1), (b,2), (a,3), (d,4) \in g$ once we have these $g \circ f = \iota_A$ irresp. of the value of g at c. There are 4 choices for $g(c)$ $\therefore$ 4

16. [4] Use the Euclidean Algorithm to find $d = gcd(3142, 900)$ and then use your work to find integers x and y such that $3142x + 900y = d$.

$3142 = 3 \times 900 + 442$
$900 = 2 \times 442 + 16$
$442 = 27 \times 16 + 10$
$16 = 1 \times 10 + 6$
$10 = 1 \times 6 + 4$
$6 = 1 \times 4 + 2$
$4 = 2 \times 2 + 0$

$\therefore gcd(3142, 900) = 2$

$2 = 6 - 4$
$= 6 - (10 - 6)$
$= 2 \times 6 - 10$
$= 2(16 - 10) - 10$
$= 2 \times 16 - 3 \times 10$
$= 2 \times 16 - 3(442 - 27 \times 16)$
$= 83 \times 16 - 3 \times 442$
$= 83(900 - 2 \times 442) - 3 \times 442$
$= 83 \times 900 - 169 \times 442$
$= 83 \times 900 - 169(3142 - 3 \times 900)$
$= 3142(-169) \leftarrow x$
$+ 900(590) \leftarrow y$

17. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

F When -120 is divided by -11, the remainder is -1.

T For all integers n, the integers n and $n + 1$ are relatively prime.

T If a and b are integers such that $\text{lcm}(a, b) < |ab|$, then $\gcd(a, b) > 1$.

T If a and d are positive integers such that $d \mid a$, then there exists an integer n such that $gcd(a, n) = d$.

18. [1] Complete this definition. Let a and b be integers. We say that a *divides* b, and write $a \mid b$, if: there exists $k \in \mathbb{Z}$ s.t. $ak = b$.

19. [3] Let a, b and d be integers such that $d \mid ab$ and $gcd(a, d) = 1$. Prove that $d \mid b$. Give an example to show that the conclusion may not hold if $gcd(a, d) > 1$.

Since $\gcd(a,d) = 1$, there exist $x, y \in \mathbb{Z}$ s.t. $ax + dy = 1$.

$\therefore abx + dby = b$

Now $d \mid ab$ by hypothesis and $d \mid db$, so $d \mid abx$ & $d \mid dby$

$\therefore d \mid \underbrace{abx + dby}_{= b}$

$6 \mid 4 \cdot 3$ but $6 \nmid 3$ and $6 \nmid 4$

Note $\gcd(6,4) > 1$

20. [3] Find the base 9 representation of 2016.

$2016 = 9 \times 224 + 0$

$224 = 9 \times 24 + 8$

$24 = 9 \times 2 + 6$

$2 = 9 \times 0 + 2$

done

$\therefore 2016 = (2680)_9$

21. [2] Use the blank to indicate whether each statement is True or False. All variables are integers. No justification is necessary.

T The last digit of 7^{201} is 7.

F There exists an integer k such that $7k \mid (144)^5$

T If $a \equiv 40 \cdot 38^9 - 5 \pmod{13}$, then $13 \mid a^2 - 10$.

$1 \; (-1) - 5 \equiv -6$

T $(11110101)_2 = (F5)_{16}$.

F 5

$7^2 = 49 \equiv -1 \pmod{10}$

$\therefore 7^4 \equiv 1 \pmod{10}$

$\therefore 7^{200} \equiv (7^4)^{50} \equiv 1 \pmod{10}$

$7^{201} = 7 \cdot 7^{200} \equiv 7 \pmod{10}$

22. [3] Let n be a positive integer. Prove that

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = n.$$

Hint: consider the cases of n even and n odd.

1st suppose n is even. $\therefore n = 2k$ for some $k \in \mathbb{Z}$

$\therefore \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{2k}{2} \right\rfloor + \left\lceil \frac{2k}{2} \right\rceil = k + k = 2k = n$ ✓

Now suppose n is odd $\therefore n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$.

$\therefore \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{2\ell+1}{2} \right\rfloor + \left\lceil \frac{2\ell+1}{2} \right\rceil$

$= \left\lfloor \ell + \frac{1}{2} \right\rfloor + \left\lceil \ell + \frac{1}{2} \right\rceil$

$= \ell + (\ell + 1) = 2\ell + 1 = n$ ✓

Since the stmt holds in both cases, it is proved

23. [4] Let $a_0, a_1, a_2, \ldots$ be the sequence defined by $a_0 = 2$, and $a_n = 3a_{n-1} + 2$ for $n \geq 1$. Find a_1, a_2, a_3 and a_4, then use your work to obtain a formula for a_n (note: a formula, not just a summation). It is not necessary prove that your formula is correct.

$a_0 = 2$

$a_1 = 3a_0 + 2 = 3 \cdot 2 + 2 = 8$

$a_2 = 3a_1 + 2 = 3(3 \cdot 2 + 2) + 2$

$= 3^2 \cdot 2 + 3 \cdot 2 + 2 = 26$

$a_3 = 3a_2 + 2 = 3(3^2 \cdot 2 + 3 \cdot 2 + 2) + 2$

$= 3^3 \cdot 2 + 3^2 \cdot 2 + 3 \cdot 2 + 2 = 80$

$a_4 = 3a_3 + 2 = 3(3^3 \cdot 2 + 3^2 \cdot 2 + 3 \cdot 2 + 2) + 2$

$= 3^4 \cdot 2 + 3^3 \cdot 2 + 3^2 \cdot 2 + 3 \cdot 2 + 2 =$ 242

Guess $a_n = 3^n \cdot 2 + 3^{n-1} \cdot 2 + \cdots + 2$

$= 2(3^n + 3^{n-1} + \cdots + 1)$

$= \cancel{2}\left(\frac{3^{n+1} - 1}{\cancel{3-1}}\right) = 3^{n+1} - 1$

24. [4] Use induction to prove that, for all $n \geq 1$,

$$1(2) + 2(3) + 3(4) + \cdots + n(n+1) = n(n+1)(n+2)/3.$$

Basis. when $n=1$, LHS $= 1(2) = 2$
& RHS $= 1(2)(3)/3 = 2$

Since LHS = RHS, stmt true when $n=1$.

IH. Assume the equality holds for $n=1, n=2, \ldots, n=k$ for some $k \geq 2$.

i.e. $1(2) = 1(1+1)(1+2)/3$

$1(2) + 2(3) = 2(2+1)(2+2)/3$

$\vdots$

$1(2) + 2(3) + \cdots + k(k+1) = k(k+1)(k+2)/3$

IS Want $1(2) + 2(3) + \cdots + (k+1)((k+1)+1) = (k+1)(k+2)(k+3)/3$

Consider the LHS

$$1(2) + 2(3) + \cdots + (k+1)(k+2)$$
$$= \underbrace{1(2) + 2(3) + \cdots + k(k+1)}_{} + (k+1)(k+2)$$
$$= k(k+1)(k+2)/3 + (k+1)(k+2) \cdot \frac{3}{3}$$
$$= \frac{(k+1)(k+2)}{3}\left[k+3\right] \text{, as wanted}$$

$\therefore$ By induction $1(2) + 2(3) + \cdots + n(n+1) = \frac{n(n+1)(n}{3}$ $\forall n \geq 1$

25. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

T If $A \times A$ has 100 elements, then A has 10 elements.

T There are $2^{(n^2)}$ relations on a set with n elements.

F If A has two elements, then there are no transitive relations on A.

F A set with 7 elements has $2^7 - 1$ different non-empty proper subsets.

26. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 1,\ b_1 = 4$ and $b_n = 4b_{n-1} - 4b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = (n+1)2^n$ for all $n \geq 0$.

Basis: when $n=0$, $b_0 = 1$ & $(0+1)\,2^0 = 1$ ✓
when $n=1$, $b_1 = 4$ & $(1+1)\,2^1 = 4$. ✓
∴ stmt true when $n=0$ & when $n=1$.

IH For some $k \geq 1$, assume
$b_0 = 1,\ b_1 = 4,\ \ldots,\ b_k = (k+1)2^k$.

IS Want $b_{k+1} = ((k+1)+1)2^{k+1}$

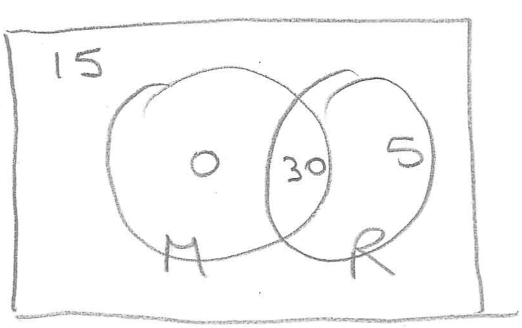
Look at b_{k+1}. Since $k \geq 1$, $k+1 \geq 2$, so

$$\begin{aligned} b_{k+1} &= 4b_k - 4b_{k-1} \\ &= 4(k+1)2^k - 4((k-1)+1)2^{k-1} \quad \text{by IH (twice)} \\ &= 2(k+1)2^{k+1} - k\,2^{k+1} \\ &= 2^{k+1}[2k+2-k] = (k+2)2^{k+1}, \text{ as wanted} \end{aligned}$$

∴ By induction $b_n = (n+1)2^n \ \forall n \geq 0$.

27. [2] A bicycle shop surveyed 50 customers and found that 30 own a mountain bike, 35 own a road bike, and 15 did not currently own a bicycle. How many of the 50 customers own both a mountain bike and a road bike?

Let M be the set of customers with a MTB
R " " " " " " road bike.



$$|M \cup R| = |M| + |R| - |M \cap R|$$
$$35 = 30 + 35 - 30$$

30 have both a mtb & a road bike.

/END