Review Questions on Relations from Section 6.11

6. Answer each question true or false, and briefly explain your reasoning.

   (a) If $|A| = 4$, then there are exactly 2^{16} relations on A.

   True. If $|A|=4$, then $|A\times A| = 4\times 4 = 16$.
   Any relation on A is a subset of $A\times A$.
   $A\times A$ has 2^{16} subsets

   (b) If $\mathcal{R}$ is an anti-symmetric relation on $\mathbb{Z}$ and $(1,2) \notin \mathcal{R}$, then $(2,1) \in \mathcal{R}$.

   False. ϕ is an anti-symmetric relation on any set.

   (c) For any set A, there is exactly one relation on A which is reflexive, symmetric, transitive and anti-symmetric.

   True. If the relation is reflexive it must contain all pairs (x,x), $x\in A$.
   It can't contain any other pairs and be both symmetric & anti-symmetric.

   (d) The relation $\sim$ on $\{2,3\}$, defined by $x \sim y$ if and only if xy is odd, is reflexive.

   False. $2\cdot 2 = 4$ is even
   $\therefore 2 \not\sim 2$
   $\therefore \sim$ is not reflexive.

   (e) The set of all relations from A to B is $\mathcal{P}(A \times B)$.

   True. A relation from A to B is a subset of $A\times B$. The power set of $A\times B$ is the set of all subsets of $A\times B$.

   (f) For the set $A = \{1,2,3\}$, if the relation $\mathcal{R}$ on A is anti-symmetric and $(1,3) \in \mathcal{R}$, then $\mathcal{R}$ is not symmetric.

   True. Since R is anti-symmetric & $(1,3)\in R$, $(3,1)\notin R$
   $\therefore R$ is not symmetric.

(g) For any set A, there is a relation $\mathcal{R}$ on A that is both symmetric and anti-symmetric.

True. $\emptyset$ is a relation A that is symmetric & anti-symmetric.

12. Suppose that $\mathcal{R}$ and $\mathcal{S}$ are relations on a non-empty set A. Determine if each of the following statements is true or false. Prove each true statement. For each false statement, give a counterexample using $A = \{1, 2, 3\}$.

(a) If $\mathcal{R}$ and $\mathcal{S}$ are both anti-symmetric, then $\mathcal{R} \setminus \mathcal{S}$ is anti-symmetric.

True. Suppose $(x,y), (y,x) \in R \setminus S$

$\therefore (x,y), (y,x) \in R$

Since R is anti-symmetric, $x = y$

$\therefore R \setminus S$ is anti-symmetric.

(b) If neither $\mathcal{R}$ nor $\mathcal{S}$ is symmetric, then $\mathcal{R} \cup \mathcal{S}$ is not symmetric.

Let $R = \{(1,2)\}$ & $S = \{(2,1)\}$

Then neither R nor S is symmetric.

But $R \cup S = \{(1,2), (2,1)\}$ is symmetric.

(c) If $\mathcal{R}$ and $\mathcal{S}$ are both equivalence relations, then so is $\mathcal{R} \cap \mathcal{S}$.

True.

<u>reflexive</u>. Take any $x \in A$.

Since R is reflexive, $(x,x) \in R$

" S " " , $(x,x) \in S$

$\therefore (x,x) \in R \cap S$ $\therefore R \cap S$ is reflexive.

symmetric : Suppose $(x,y) \in R \cap S$
$\therefore (x,y) \in R$ and $(x,y) \in S$
Since R & S are symmetric
$(y,x) \in R$ and $(y,x) \in S$
$\therefore (y,x) \in R \cap S$ $\therefore$ $R \cap S$ is symmetric

transitive Suppose $(x,y),(y,z) \in R \cap S$,
$\therefore (x,y),(y,z) \in R$ and $(x,y),(y,z) \in S$
Since R & S are transitive $(x,z) \in R$
and $(x,z) \in S$ $\therefore (x,z) \in R \cap S$
$\therefore$ $R \cap S$ is transitive

$\therefore$ $R \cap S$ is an equivalence relation.

23. Take it as given that the relation $\mathcal{R}$ on $A = \{1, 2, \ldots, 46\}$ defined by $x\mathcal{R}y$ if and only if $x - y$ is a multiple of 10 is an equivalence relation.

(a) How many of the equivalence classes [6], [13], [16], [28], [38], [46] are different? Why? Explain in at most two sentences.

3. We know $[6] = [16] = [46]$ b/c any two of 6, 16, 46 are related. Similarly $[28] = [38]$. No two of 6, 13 & 28 are related, so $[6], [28], [13]$ are all disjoint.

(b) How many subsets belong to the partition of A determined by $\mathcal{R}$? Why?

Ten. Two numbers are related if they have the same last digit, and there are 10 possibilities.