SOLUTION

[2] 1. Check the appropriate circle to indicate whether each statement is true or false. You do not need to provide any reasoning.

	True	False	
(a)	○	⊘	If the set S contains 6 elements, then S has $2^6 = 64$ proper subsets.
(b)	⊘	○	Let A and B be sets. If $A \cap B = \varnothing$, then $\lvert A \cup B\rvert = \lvert A\rvert + \lvert B\rvert$.
(c)	○	⊘	In the inductive hypothesis to prove a statement $S(n)$ using weak induction, we assume $S(k)$ holds for all k.
(d)	⊘	○	If a sequence is recursively defined by $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$ where $a_1 = 2$ and $a_2 = -1$, then $a_4 = 0$.

[2] 2. In a group of 130 people surveyed, 47 have a cat, 65 have a dog, and 32 have both a cat and a dog. How many people have neither a cat nor a dog?

$|\mathcal{U}| = 130, \ |C| = 47, \ |D| = 65, \ |C \cap D| = 32$

$|C \cup D| = 47 + 65 - 32 = 80$

$|(C \cup D)^c| = |\mathcal{U}| - |C \cup D| = 130 - 80 = 50$

so 50 people have neither

[3] 3. Consider the recursive sequence $a_n = a_{n-1} + 4n$ for $n \geq 1$ with $a_0 = 7$.

(a) Find a_1, a_2, a_3, and a_4. You may leave your answers unsimplified.

$a_0 = 7$

$a_1 = 7 + 4 \cdot 1$

$a_2 = (7 + 4 \cdot 1) + 4 \cdot 2$

$a_3 = (7 + 4 \cdot 1 + 4 \cdot 2) + 4 \cdot 3$

$a_4 = (7 + 4 \cdot 1 + 4 \cdot 2 + 4 \cdot 3) + 4 \cdot 4$

(b) Use your work in (a) to conjecture (guess) a formula for a_n that depends only on n, and is valid for all $n \geq 0$. Your answer should not involve a sum of many terms. You do not need to prove that your conjecture is correct.

$a_n = 7 + 4 \cdot 1 + 4 \cdot 2 + \cdots + 4 \cdot n$

$= 7 + 4(1 + 2 + \cdots + n)$

$= 7 + 4 \cdot \frac{n(n+1)}{2} = 7 + 2n(n+1)$

[3] 4. Let A and B be sets. Using any method, prove that $(A \cap B^c) \cup B = A \cup B$.
Hint: There is short proof using the laws of set theory.

$$(A \cap B^c) \cup B = (A \cup B) \cap (B^c \cup B) \quad \text{distributivity}$$
$$= (A \cup B) \cap \mathcal{U} \quad \text{known equality}$$
$$= A \cup B \quad \text{identity}$$

[5] 5. Use induction to prove that $1 + 5 + 5^2 + 5^3 + \cdots + 5^n = \frac{1}{4}(5^{n+1} - 1)$ for all integers $n \geq 0$.

Base case when $n=0$, LHS$=1$, RHS$= \frac{1}{4}(5^1 - 1) = \frac{1}{4} \cdot 4 = 1$ ✓

Inductive hyp
(weak) Suppose $k \in \mathbb{Z}$, $k \geq 0$, and assume the statement is true when $n=k$, so that $1 + 5 + 5^2 + \cdots + 5^k = \frac{1}{4}(5^{k+1} - 1)$.

OR

(strong) Suppose $k \in \mathbb{Z}$, $k \geq 0$, and assume the statement is true for all $n \in \mathbb{Z}$ with $0 \leq n \leq k$, so that $1 + 5 + 5^2 + \cdots + 5^n = \frac{1}{4}(5^{n+1} - 1)$.

Inductive step Want to show $1 + 5 + 5^2 + \cdots + 5^{k+1} = \frac{1}{4}(5^{k+2} - 1)$.

$$1 + 5 + 5^2 + \cdots + 5^k + 5^{k+1} = \frac{1}{4}(5^{k+1} - 1) + 5^{k+1} \quad \text{(by IH)}$$
$$= \frac{5^{k+1} - 1}{4} + \frac{4 \cdot 5^{k+1}}{4}$$
$$= \frac{5^{k+1} + 4 \cdot 5^{k+1} - 1}{4}$$
$$= \frac{5 \cdot 5^{k+1} - 1}{4} = \frac{5^{k+2} - 1}{4}, \text{ as needed}$$

Conclusion By PMI, $1 + 5 + 5^2 + \cdots + 5^n = \frac{1}{4}(5^{n+1} - 1)$ for all $n \in \mathbb{Z}$, $n \geq 0$.