201501 Math 122 [A03] Midterm #2

#V00:________________ Name: Solutions

This midterm has 4 pages and 10 questions. There are 30 marks available. The time limit is 50 minutes. Calculators will not help with these questions. Except when indicated, it is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the pages for rough or extra work.

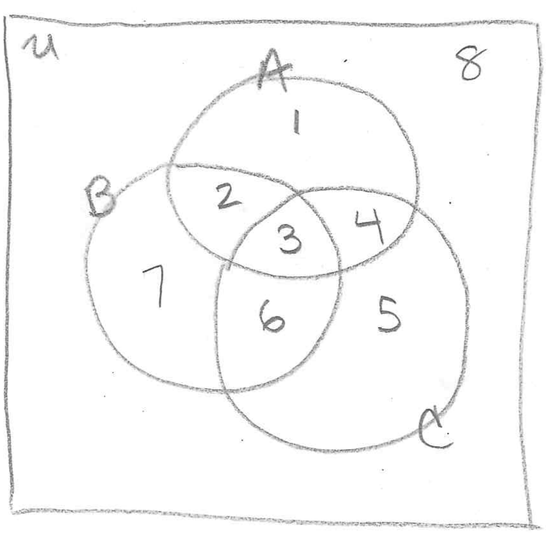
1. [3] Let A and B be sets. Prove that $(A \cap B)^c = A^c \cup B^c$.

We have
$$\begin{aligned}
& x \in (A \cap B)^c \\
\iff & x \notin A \cap B \\
\iff & (x \notin A) \vee (x \notin B) \\
\iff & (x \in A^c) \vee (x \in B^c) \\
\iff & x \in A^c \cup B^c
\end{aligned}$$

2. [3] Give a counterexample to show that the statement

$$(A \oplus B) \cap C^c = (A \cap C^c) \oplus B, \textit{ for all sets } A, B, \textit{ and } C$$

is false.



$\mathcal{U} = \{1, 2, \ldots, 8\}$

$A = \{1,2,3,4\}$, $B = \{2,3,6,7\}$,

$C = \{3,4,5,6\}$, $C^c = \{1,2,7,8\}$

$A \oplus B = \{1,4,6,7\}$

$\underline{\text{LHS}} = \{1,7\}$

$A \cap C^c = \{1,2\}$

$\underline{\text{RHS}} = \{1,36,7\}$

$\text{LHS} \neq \text{RHS}.$

3. [3] Let A, B and C be sets. Prove that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

Let $(x,y) \in A \times (B \cap C)$.

Then $x \in A$ and $\underbrace{y \in B \cap C}$

$\therefore y \in B \wedge y \in C$.

$\therefore (x,y) \in A \times B \wedge (x,y) \in A \times C$

$\therefore (x,y) \in (A \times B) \cap (A \times C)$.

4. [3] Let $\sim$ be a reflexive, symmetric and transitive relation on $A = \{1,2,3\}$ such that $1 \sim 2$ and $3 \not\sim 2$. Write $\sim$ as a set of ordered pairs.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Refl $\Rightarrow$ $1 \sim 1$, $2 \sim 2$, $3 \sim 3$

Symm $\Rightarrow$ $2 \sim 1$, $2 \not\sim 3$

If $3 \sim 1$ then $3 \sim 2$ by transitivity $\Rightarrow \Leftarrow$

$\therefore 3 \not\sim 1$ and $1 \not\sim 3$

$\therefore \sim = \{(1,1), (2,2) (3,3), (1,2), (2,1)\}$

5. [2] Use the blank to indicate whether each statement is true or false. No reasons are necessary.

_T_ For all sets A and B, $A \cap (A \cup B)^c = \emptyset$.

_F_ If $A \times B \neq \emptyset$ then $A \neq \emptyset$ and $B \neq \emptyset$.

_T_ The set of all relations from A to B is $\mathcal{P}(A \times B)$.

_T_ The relation $\sim$ on $\{1,2\}$, defined by $x \sim y$ if and only if xy is odd, is anti-symmetric.

6. [3] Let S be the set of all squares drawn in the plane with one side along the x-axis and bottom left corner at $(0,0)$. Let $\mathcal{R}$ be the relation on S defined by $s_1 \mathcal{R} s_2$ if and only if the perimeter of s_1 is at least as large as the perimeter of s_2, Prove that $\mathcal{R}$ is anti-symmetric.

Suppose $s_1 R s_2$ and $s_2 R s_1$.
Then perm $s_1 \geq$ perm s_2
& perm $s_2 \geq$ perm s_1
$\therefore$ perm s_1 = perm s_2.
$\therefore$ The sides of s_1 & s_2 are of the same length
$\therefore s_1 = s_2$

7. (a) [3] Let $\sim$ be an equivalence relation on the set A. Suppose $x, y \in A$ and $x \not\sim y$. Prove that $[x] \cap [y] = \emptyset$. (Hint, assume there exists $z \in [x] \cap [y]$ and derive a contradiction.)

Suppose $z \in [x] \cap [y]$.
Then $z \sim x$ and $z \sim y$.
By symmetry, $x \sim z$.
By transitivity $x \sim y \Rightarrow \Leftarrow$.
$\therefore [x] \cap [y] = \phi$.

(b) [2] The relation $\mathcal{R}$ b on $A = \{1, 2, \ldots, 42\}$ defined by $x \mathcal{R} y$ if and only if $x - y$ is a multiple of 10 is known to be an equivalence relation (you do not need to prove it!). How many of the equivalence classes $[12], [39], [42], [24], [35], [29], [15]$ are different? Why?

4 $\because$ $[x] = [y] \Leftrightarrow xRy$
$\Leftrightarrow$ x & y have the same last digit.

8. [2] Use the blank to indicate whether each statement is true or false. No reasons are necessary.

F If $A = \{w, x, y, z\}$ and $B = \{1, 2, 3, 4, 5\}$ then $\{(w, 1), (x, 2), (y, 3), (z, 4), (w, 5)\}$ is a function from A to B.

T The integer n is even if and only if $2 \lfloor \frac{n}{2} \rfloor = n$

T If f and g are functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x) = x^2 - 4$ and $g(x) = (x-2)(x+2)$ then $f = g$.

F For every function $f : A \to B$, the set $\{(y, x) : (x, y) \in f\}$ is a function from B to A.

9. Let $f : (0, \infty) \to \mathbb{R}$ be the function defined by $f(x) = x^2$.

(a) [2] Prove that f is 1-1.

Suppose $f(x_1) = f(x_2)$.

Then $x_1^2 = x_2^2$

$\therefore$ $\sqrt{x_1^2} = \sqrt{x_2^2}$

i.e. $|x_1| = |x_2|$.

But $x_1, x_2 > 0$, so $x_1 = x_2$.

$\therefore$ f is 1-1.

(b) [2] Explain why f is not onto.

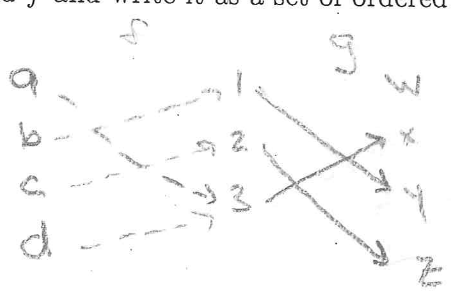
We always have $f(x) > 0$.

There is no x s.t. $f(x) = -1$.

Since -1 is in the target,

f is not onto.

10. [2] Let $f : A \to B$ and $g : B \to C$ be functions with $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$, $C = \{w, x, y, z\}$ such that $g \circ f = \{(a, x), (b, y), (c, z), (d, x)\}$ and $g = \{(1, y), (2, z), (3, x)\}$. Find f and write it as a set of ordered pairs.



$f = \{(a,3), (b,1), (c,2), (d,3)\}$