

[A03]

## 201501 Math 122 [A02] Midterm #3

#V00: \_\_\_\_\_

Name: Solutions

This midterm has 4 pages and 11 questions. There are 30 marks available. The time limit is 50 minutes. Only the Sharp EL-510R calculator is allowed. Except when explicitly noted, it is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the pages for rough or extra work.

1. [2] Find
- $b$
- if
- $(123)_b = (150)_6$
- .

$$(150)_6 = 1 \cdot 6^2 + 5 \cdot 6 + 0 = 66$$

$$(123)_b = b^2 + 2b + 3$$

$$b^2 + 2b + 3 = 66 \Leftrightarrow b^2 + 2b - 63 = 0$$

$$\Leftrightarrow (b+9)(b-7) = 0 \Leftrightarrow b = -9 \text{ or } b = 7.$$

$-9$  is not a base  $\therefore$   $b = 7$

2. Let
- $a, b \in \mathbb{Z}$
- .

- (a) [2] Suppose there exist integers
- $x$
- and
- $y$
- such that
- $ax + by = 1$
- . Prove that
- $\gcd(a, b) = 1$
- .

$$\gcd(a, b) \mid a \text{ and } \gcd(a, b) \mid b$$

$$\therefore \gcd(a, b) \mid ax + by = 1.$$

$$\text{Since } \gcd(a, b) \geq 1, \gcd(a, b) = 1.$$

- (b) [2] Suppose there are integers
- $x$
- and
- $y$
- such that
- $ax + by = 9$
- . What are the possible values of
- $\gcd(a, b)$
- ? Explain.

$$\text{As } m \mid a, \gcd(a, b) \mid 9.$$

$$\text{Since } \gcd(a, b) \geq 1, \text{ it equals } 1, 3 \text{ or } 9.$$

3. [2] Use the Fundamental Theorem of Arithmetic (Unique Factorization Theorem) to explain why there are no positive integers
- $k$
- and
- $n$
- such that
- $5k = 3^n$
- .

By the FTA, the only prime factors of the LHS are those on the RHS, namely 3s.

4. [2] Use the blank to indicate whether each statement is true or false. No reasons are necessary.

T If  $f: A \rightarrow B$  and  $g: B \rightarrow C$  are each invertible then so is  $g \circ f$ .

T If  $f: A \rightarrow B$  is a function, then  $(\iota_B \circ f) \circ \iota_A = f$ .

T If  $f: X \rightarrow Y$  is a one-to-one correspondence then so is  $f^{-1}: Y \rightarrow X$ .

T Suppose  $f: A \rightarrow B$  and  $g: B \rightarrow C$ . If  $f$  is not 1-1, then  $g \circ f$  is not 1-1.

5. Suppose  $n = (d_2d_1d_0)_5$ .

(a) [2] Prove that  $n \equiv d_2 + d_1 + d_0 \pmod{2}$ .

$$\begin{aligned}n &= d_2 \cdot 5^2 + d_1 \cdot 5 + d_0 \\ &\equiv d_2 \cdot 1^2 + d_1 \cdot 1 + d_0 \pmod{2} \\ &\text{since } 5 \equiv 1 \pmod{2}\end{aligned}$$

(b) [2] Is it true that  $n$  is even if and only if  $d_2 + d_1 + d_0$  is even? Explain.

$$\begin{aligned}\text{Yes: } n \text{ is even} &\Leftrightarrow n \equiv 0 \pmod{2} \\ &\Leftrightarrow d_2 + d_1 + d_0 \equiv 0 \pmod{2} \\ &\Leftrightarrow d_2 + d_1 + d_0 \text{ is even}\end{aligned}$$

6. [4] Use the Principle of Mathematical Induction to prove that, for all integers  $n \geq 2$ , any postage of  $n$  cents can be made using 2 cent and 3 cent stamps.

Basis Since postages of 2¢ and 3¢ can each be made with a single stamp, the stmt is true when  $n=2$  and when  $n=3$ .

IH. Suppose each postage of 2, 3, ...,  $k$  cents can be made with 2¢ and 3¢ stamps, for some  $k \geq 3$ .

IS To make  $(k+1)$ ¢ postage, first make  $(k-1)$ ¢ and then add a 2¢ stamp. The postage  $(k-1)$ ¢ can be made by the IH b/c  $k \geq 3$  and so  $k-1 \geq 2$ .

$\therefore$  By induction,  $\forall n \geq 2$  any postage of  $n$  cents can be made with 2¢ & 3¢ stamps.

7. [2] Use the blank to indicate whether each statement is true or false. No reasons are necessary.

F  $(2837)_8$  is the base-8 representation of some integer.

F  $2^{80}3^{90}5^{30} \mid 2^{90}3^{80}5^{30}$ .

F If  $x \equiv 5 \pmod{8}$  and  $y \equiv 6 \pmod{8}$  then  $xy \equiv 3 \pmod{8}$ .

T If  $a^4 \equiv 9 \pmod{10}$  then the last digit of  $a^8$  is 1.

8. [4] Let  $a_0, a_1, \dots$  be the sequence recursively defined by  $a_0 = 1$ , and  $a_n = 6a_{n-1} + 1$ . Use the Principle of Mathematical Induction to prove that  $a_n = \frac{6^{n+1} - 1}{5}$  for all integers  $n \geq 0$ .

Basis: When  $n=0$ , we have  $a_0 = 1 = \frac{6^{0+1} - 1}{5} = \frac{5}{5}$ .  
 $\therefore$  The stmt is true when  $n=0$ .

IH: Suppose  $a_k = \frac{6^{k+1} - 1}{5}$  for some  $k \geq 0$ .

IS: Consider  $a_{k+1}$ . Since  $k+1 \geq 1$ ,

$$\begin{aligned} a_{k+1} &= 6a_k + 1 = 6 \left( \frac{6^{k+1} - 1}{5} \right) + 1 \quad (\text{by IH}) \\ &= \frac{6 \cdot 6^{k+1} - 6 + 5}{5} \\ &= \frac{6^{k+2} - 1}{5}, \text{ as needed.} \end{aligned}$$

$\therefore$  By induction,  $a_n = \frac{6^{n+1} - 1}{5} \quad \forall n \geq 0$ .

9. [2] Use the blank to classify each set as countable or uncountable. No reasons are necessary.

Countable  $\{ \lfloor x/3 \rfloor : 0 \leq x \leq 7 \}$ .

Uncountable  $\{(x, y) : y = 4x + 1, x, y \in \mathbb{R}\}$ .

Countable  $\{k + \sqrt{2} : k \in \mathbb{N}\}$ .

uncountable The set of lines in the plane that pass through the point  $(0, 0)$ .

10. Suppose you are given a list of infinite sequences of 0s and 1s:

1. 0, 0, 1, 0, ...

2. 1, 1, 1, 0, ...

3. 0, 1, 1, 1, ...

$\vdots$

Let  $\mathcal{L} = \ell_1, \ell_2, \dots$  be the sequence defined by

$$\ell_i = \begin{cases} 1 & \text{if element } i \text{ of sequence } i \text{ is 0} \\ 0 & \text{if element } i \text{ of sequence } i \text{ is 1.} \end{cases}$$

(a) [1] Write down  $\ell_1, \ell_2$  and  $\ell_3$ .

$$\ell_1 = 1, \quad \ell_2 = 0, \quad \ell_3 = 0$$

(b) [2] Can the sequence  $\mathcal{L}$  appear anywhere in the list? Explain.

No. By definition, it differs from sequence  $i$  in element  $i$ , for all  $i$ .

11. [1] Let  $A$  and  $B$  be finite sets. Fill in each blank. No reasons are necessary.

(a)  $|A \cup B| = \underline{|A| + |B| - |A \cap B|}$

(b)  $|A \times B| = \underline{|A| \times |B|}$