201601 Math 122 [A01] Midterm #2

#V00:_____ Name: Solutions

This midterm has 4 pages and 10 questions. There are 30 marks available. The time limit is 50 minutes. Calculators will not help with these questions. Except when indicated, it is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the pages for rough or extra work.

1. [2] Give a counterexample to show that the statement

$$(A \setminus B) \cup C^c = (A \cup B) \setminus C, \textit{ for all sets } A, B, \textit{ and } C$$

is false. Let $\mathcal{U} = \{1,2,3\}$.

Let $A = B = C = \phi$.

$LHS = (\phi \setminus \phi) \cup \phi^c = \mathcal{U} = \{1,2,3\}$.

$RHS = (\phi \cup \phi) \setminus \phi = \phi$

$LHS \neq RHS$

$\therefore$ The statement is false.

2. [2] Let A, B, C and D be sets such that $A \subseteq C$ and $B \subseteq D$. Prove that $A \times B \subseteq C \times D$.

Let $(x,y) \in A \times B$. Then $x \in A$ and $y \in B$.

Since $A \subseteq C$, $x \in C$. Since $B \subseteq D$, $y \in D$.

$\therefore (x,y) \in C \times D$.

$\therefore A \times B \subseteq C \times D$

3. [2] Let $A = \{1, \{2,3\}, \{1,2,3\}\}$ and $B = \{1, 2, \{1,2,3\}\}$. Use the blank to indicate whether each statement is true or false. No reasons are necessary.

F $A \subseteq B$.

F $\{1,2,3\} \in \mathcal{P}(A)$.

T $\emptyset \subsetneq A$.

F $A \cap B$ has exactly 3 different elements.

4. [3] Let A and B be sets. Use any method to show that $(A \oplus B)^c = (A^c \cup B) \cap (A \cup B^c)$.

$$(A \oplus B)^c = ((A \setminus B) \cup (B \setminus A))^c \quad \text{Def'n.}$$
$$= ((A \cap B^c) \cup (B \cap A^c))^c \quad \text{Known.}$$
$$= (A \cap B^c)^c \cap (B \cap A^c)^c \quad \text{DeMorgan}$$
$$= (A^c \cup B) \cap (B^c \cup A) \quad \text{DeMorgan \& Dbl Comp}$$
$$= (A^c \cup B) \cap (A \cup B^c) \quad \text{Comm.}$$

5. [3] Let $\mathcal{R}$ be a reflexive, anti-symmetric and transitive relation on $A = \{1, 2, 3\}$ such that $2 \mathcal{R} 1$ and $1 \mathcal{R} 3$. Write $\mathcal{R}$ as a set of ordered pairs.

$$R = \{(1,1), (2,2), (3,3), (2,1), (1,3), (2,3)\}$$

refl, given, trans.

Anti-symmetry $\Rightarrow (1,2), (3,1), (3,2) \notin R$

6. Each statement below is false. In each case, give a counterexample to the given statement, or a reason why it is false.

(a) [1] For sets A and B, if $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$, then $A \cap B = \emptyset$.

Let $A = \{1, 2\}$ & $B = \{2, 3\}$. Then $A \setminus B = \{1\} \neq \emptyset$ $B \setminus A = \{3\} \neq \emptyset$ and $A \cap B = \{2\} \neq \emptyset$

(b) [1] If $A \times B = B \times A$ then $A = B$.

Let $A = \emptyset$ and $B = \{1\}$. Then $A \times B = \emptyset = B \times A$ but $A \neq B$.

(c) [1] If a relation on $A = \{1, 2\}$ is symmetric, then it can not be anti-symmetric.

$R = \{(1,1)\}$ is symmetric and anti-symmetric.

(d) [1] If $f : \{1, 2\} \to \{1, 2, 3\}$ and $g : \{1, 2, 3\} \to \{1, 2\}$ are such that $g \circ f(x) = x$ for every $x \in \{1, 2\}$, then g is the inverse of f.

f can not have an inverse as it can not be onto.

7. Let $\sim$ be the relation on $T = \{10, 11, \ldots, 99\}$ defined by $a \sim b \Leftrightarrow a$ has the same last digit as b (that is, the same rightmost digit as b; for example, $24 \sim 94$).

   (a) [4] Prove that $\sim$ is an equivalence relation.

reflexive. Any number $x \in T$ has the same last digit as itself. $\therefore x \sim x \quad \forall x \in T$.

symmetric. Suppose $x \sim y$. Then x has the same last digit as y.
$\therefore y$ has the same last digit as x
$\therefore y \sim x$.

transitive. Suppose $x \sim y$ and $y \sim z$.
Then x has the same last digit as y,
and y has the same last digit as z.
$\therefore x$ has the same last digit as z.
$\therefore x \sim z$

$\therefore \sim$ is an equivalence relation.

   (b) [1] How many of the equivalence classes [12], [39], [42], [24], [94], [74], [15] are different? Why? Explain in one sentence.

4. There are 4 diffent last digits

   (c) [1] How many subsets belong to the partition of T induced by $\sim$? Why?

10. One for each possible last digit

8. [2] Use the blank to indicate whether each statement is true or false. No reasons are necessary.

   F The relation $\{(x, y) : x^2 + y^2 = 1\}$ is a function from $\mathbb{R}$ to $\mathbb{R}$.

   T The integer n is odd if and only if $2 \times \left\lceil \frac{n}{2} \right\rceil = n + 1$.

   F If $x \in \mathbb{R}$, then $\lceil x \rceil = 1 + \lfloor x \rfloor$.

   F There is a function $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4, 5, 6\}$ that contains exactly five ordered pairs.

9. [4] Let (a, b) denote the open interval of real numbers between a and b, and let $f : (1, 4) \to (1, 2)$ be defined by $f(x) = 1 + \frac{x-1}{3}$. Prove that f is both 1-1 and onto.

1-1. Suppose $f(x_1) = f(x_2)$

Then $1 + \frac{x_1 - 1}{3} = 1 + \frac{x_2 - 1}{3}$

$\therefore \quad \frac{x_1 - 1}{3} = \frac{x_2 - 1}{3}$

$\therefore \quad x_1 - 1 = x_2 - 1$, so $x_1 = x_2$

$\therefore$ f is 1-1.

onto. Take any $y \in (1, 2)$.

Suppose $f(x) = y$.

Then $1 + \frac{x-1}{3} = y$,

so $x = 1 + 3(y-1) = 3y - 2$.

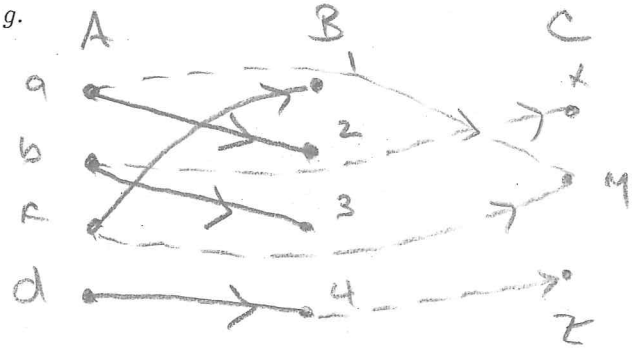
Since $1 < y < 2$ we have

$1 = 3 \cdot 1 - 2 < x < 3 \cdot 2 - 2 = 4$

$x = 3y - 2$

$\therefore \quad x \in (1, 4)$, and f is onto

10. [2] Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4\}$, and $C = \{x, y, z\}$, and let $f : A \to B$ and $g : B \to C$ be functions such that $f = \{(a, 2), (b, 3), (c, 1), (d, 4)\}$ and $g \circ f = \{(a, y), (b, x), (c, y), (d, z)\}$. Find g.



To complete the diagram, g must be

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ y & y & x & z \end{pmatrix}$$