201601 Math 122 [A01] Midterm #3

#V00:________________ Name: Key

This midterm has 4 pages and 12 questions. There are 30 marks available. The time limit is 50 minutes. Only the Sharp EL-510R or RNB calculator is allowed. Except when explicitly noted, it is necessary to show clearly organized work in order to receive any credit. Use the back of the pages for rough or extra work.

1. Suppose that $f : A \to B$ is a 1-1 function.

   (a) [1] Explain why $|rng(f)| = |A|$.

   f maps A onto $rng(f)$. Since f is 1-1, there is a 1-1 correspondence b/w A & $rng(f)$.

   (b) [1] Suppose A is uncountable. Use part (a) to show that B is uncountable.

   By (a), B has an uncountable subset, namely $rng(f)$.

   $\therefore B$ is uncountable

2. [2] Find the base-11 representation of 4110. Use the symbol A for the digit corresponding to ten.

$$4110 = 373 \times 11 + 7$$
$$373 = 33 \times 11 + 10$$
$$33 = 3 \times 11 + 0$$
$$3 = 0 \times 11 + 3$$

$\therefore 4110 = (30A7)_{11}$

3. [2] Find the integers a, b and c if $2^a 3^7 = 2^4 3^b 5^c$. Which theorem are you using in your answer?

   By the Unique Factorization Theorem, it must be the same collection of primes on the LHS & RHS

   $\therefore a = 4$, $b = 7$ and $c = 0$

4. [2] Use the blank to classify each set as Countable or Uncountable. No justification is necessary.

   C $\{a \in \mathbb{Z} : a \mid 100\}$

   U The set of linear equations of the form $y = mx + 3$, where $m \in \mathbb{R}$:

   C The set of all circles in the plane centred at $(0, 0)$ and with radius belonging to $\mathbb{Q}$.

   U $\mathcal{P}(\mathbb{N})$, the power set of the set of natural numbers:

5. [2] Use the Euclidean Algorithm to find $d = gcd(100, 46)$, and then use your work to find integers x and y such that $100x + 46y = d$.

$100 = 2 \times 46 + 8$
$46 = 5 \times 8 + 6$
$8 = 1 \times 6 + 2 \leftarrow$
$6 = 3 \times 2 + 0$
$\therefore gcd(100, 46) = 2$

$2 = 8 - 6$
$= 8 - (46 - 5 \cdot 8)$
$= 6 \cdot 8 - 46$
$= 6(100 - 2 \cdot 46) - 46$
$= 100\underbrace{(6)}_{x} + 46\underbrace{(-13)}_{y}$

6. [2] Suppose that a and b are integers such that $ab = 2^{12}3^8$. Find x and y if $gcd(a, b) = 2^5 3^x$ and $lcm(a, b) = 2^y 3^4$

$5 + y = 12 \quad \therefore y = 7$
$4 + x = 8 \quad \therefore x = 4$

7. Let $n \geq 2$ be a positive integer.

(a) [1] Suppose that $p \leq n$ is a prime number. What is the remainder when $n! + 1$ is divided by p?

$n! + 1 = \frac{n!}{p} p + 1$, and $\frac{n!}{p} \in \mathbb{Z}$. $\therefore$ the rem. is $\underline{1}$

(b) [1] Explain why part (a) implies that $n! + 1$ has a prime divisor greater than n.

It has a prime divisor, and is not divisible by any prime $\leq n$.

(c) [1] Explain why part (b) implies that there are infinitely many prime numbers.

By (b), for any $n \in \mathbb{N}$, there is a prime greater than n.

8. [2] Use the blank to indicate whether each statement is true or false. No reasons are necessary.

_F_ $|(1, 4)| = 2$.
_T_ If $gcd(a, b) = 2$ then there exist integers x and y such that $ax + by = 6$.
_F_ If d is an integer such that $d \mid ab$, then $d|a$ or $d|b$.
_T_ If $b > 2$ then $(121)_b$ is the square of an integer.

9. [2] Let n be a positive integer. Use congruences to show that $n(n+7)$ is divisible by 2.

If n is even, then $n \equiv 0 \pmod 2$,
so $n(n+7) \equiv 0(n+7) \equiv 0 \pmod 2$

If n is odd, then $n \equiv 1 \pmod 2$,
so $n(n+7) \equiv 1(1+7) \equiv 1 \cdot 0 \equiv 0 \pmod 2$.

In either case, $n(n+7) \equiv 0 \pmod 2$
$\therefore$ it is divisible by 2

10. [4] Use the Principle of Mathematical Induction to prove that

$$1(2) + 2(3) + 3(4) + \cdots + n(n+1) = n(n+1)(n+2)/3$$

for all integers $n \geq 1$.

Basis: When $n = 1$, $LHS = 1(2) = 2$
& $RHS = 1(2)(3)/3 = 2$

$\therefore$ Stmt true when $n = 1$.

IH Assume $1(2) + 2(3) + \cdots + k(k+1)$
$= k(k+1)(k+2)/3$

for some $k \geq 1$.

IS Want: $1(2) + 2(3) + \cdots + (k+1)(k+2)$
$= (k+1)(k+2)(k+3)/3$.

$LHS = \underbrace{1(2) + 2(3) + \cdots + k(k+1)} + (k+1)(k+2)$

$= \dfrac{k(k+1)(k+2)}{3} + (k+1)(k+2)$ by IH

$= \dfrac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$

$= (k+1)(k+2)(k+3)/3$, as required.

$\therefore$ By PMI, $1(2) + 2(3) + \cdots + n(n+1)$
$= n(n+1)(n+2)/3$, $\forall n \geq 1$

11. Let $a_1, a_2, \ldots$ be the sequence recursively defined by

$$a_1 = 1, \quad \text{and} \quad a_n = a_{n-1} + n2^{n-1}, \text{ for } n \geq 2.$$

(a) [1] Use the recursive definition to show that $a_4 = 49$. Credit will only be granted if you show your work.

$a_2 = 1 + 2 \cdot 2^1 = 5$
$a_3 = 5 + 3 \cdot 2^2 = 17$
$a_4 = 17 + 4 \times 2^3 = 49$

(b) [4] Use the Principle of Mathematical Induction to show that $a_n = (n-1)2^n + 1$ for all $n \geq 1$.

Basis: When $n=1$, $a_1 = 1 = (1-1)2^1 + 1$
$\therefore$ Stmt true when $n=1$.

IH: Assume $a_k = (k-1)2^k + 1$ for some $k \geq 1$.

IS: Want $a_{k+1} = k2^{k+1} + 1$.

Look at a_{k+1}. Since $k+1 \geq 2$,

$a_{k+1} = a_k + (k+1)2^{(k+1)-1}$
$= (k-1)2^k + 1 + (k+1)2^k$, by IH.
$= 2k2^k + 1$
$= k2^{k+1} + 1$, as needed

$\therefore$ By PMI, $a_n = (n-1)2^n + 1$, $\forall n \geq 1$

12. [2] Use the blank to indicate whether each statement is true or false. No reasons are necessary.

F If $k \equiv 3 \pmod 5$, then when $3k-1$ is divided by 5, the remainder is 4.
T $(2503)_7 \equiv 3 \pmod 7$.
F If $x \equiv 4 \pmod 8$ and $y \equiv 5 \pmod 8$ then $xy \equiv 1 \pmod 8$.
T If $a^2 \equiv 9 \pmod{10}$ then the last digit of a^6 is 9.