202109 Term Test 1
Math 122 A05
Instructor: Natasha Morrison

Please write your V number without the "V"
Do not open the booklet before you are told to

Date and Time: Friday, October 22, 2021 at 13.30pm.

Instructions: There are 10 pages and 8 questions. There are 24 marks available. The time limit is 45 minutes. Math and Stats standard calculators are allowed. Except when indicated, it is necessary to show clearly organized work in order to receive full or partial credit.
Please count your pages and report any discrepancy immediately to the invigilator.

True/False Instructions: Question 1 consists of 12 true/false questions labelled **TF 1** to **TF 12**. The last page of your test booklet is a bubble sheet for answering them. You can detach the back page from the rest of the test. Only fill in a bubble for questions 1-12 on the bubble sheet. When making your selection, **True is A** and **False is B**. Do not select C, D or E. Do not select more than one bubble.

Do not use the back of the bubble sheet for rough work.

Nothing written on this page will be graded.

1. [6] Use the bubble sheet provided on the last page of the test booklet to indicate whether each statement is **True (A)** or **False (B)**.

F **[TF 1]** p and $\neg(\neg p)$ can have different truth values.

T **[TF 2]** If $s_1 \to s_2$ is false, then s_1 is true.

F **[TF 3]** $a \vee (b \wedge c)$ is logically equivalent to $(a \wedge b) \vee c$.

F **[TF 4]** A valid argument has a counterexample.

F **[TF 5]** The statements $\forall y \in \mathbb{Z}, \exists x \in \mathbb{R}, x = 2y$ and $\exists x \in \mathbb{R}, \forall y \in \mathbb{Z}, x = 2y$ have the same truth value. ok

F **[TF 6]** The statement "*every integer is even*" can be written in symbols as $\exists n \in \mathbb{Z}, \forall k \in \mathbb{Z}, n = 2k$.

T **[TF 7]** $\exists x \in \mathbb{R}, (x > 1) \to (x < 1)$.

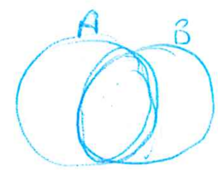
T **[TF 8]** The statements $\forall x \in \{1, 2, 3\}, x > 2$ and $(1 > 2) \wedge (2 > 2) \wedge (3 > 2)$ are logically equivalent. ok

F **[TF 9]** $b \in \{a, \{b\}, \{a, b, c\}\}$.

F **[TF 10]** If $A \cap B = A \cup B$ then $A = \emptyset$.

T **[TF 11]** $(A \cup B) \setminus (A \cap B) = A \oplus B$.

T **[TF 12]** $\emptyset \in \mathcal{P}(\{1\})$.



2. [2] Is the statement $(q \to q) \to q$ a tautology, a contradiction, or neither? Explain.

q	$q \to q$	$(q \to q) \to q$
0	1	0
1	1	1

It is neither because it ~~both~~ can be both true and false, depending on the truth value of q.
A tautology is always true and a contradiction is always false.

3. [4] Write the argument below in symbolic form, and then use known logical equivalences and inference rules to show it is valid. Remember to define the letters you use to represent statements.

The statement that I am wearing socks or I play with my puppy is false
I either play with my puppy or watch Jeopardy
If I don't go running, then I don't watch Jeopardy

$\therefore$ I go running

The argument is

$$\begin{array}{c} \neg(s \vee p) \\ p \vee j \\ \neg r \to \neg j \\ \hline \therefore r \end{array}$$

s: I am wearing socks
p: I play with my puppy
j: I watch Jeopardy
r: I go running.

(1)	$\neg(s \vee p)$	(premise)
(2)	$\neg s \wedge \neg p$	(De morgens from 1)
(3)	$p \vee j$	(premise)
(4)	$\neg p \to j$	(L.E. to (3))
(5)	$\neg r \to \neg j$	(premise)
(6)	$j \to r$	(contrapositive of (5))
(7)	$\neg p \to r$	(chain rule, 4 + 6)
(8)	$\neg p$	(from 2)
(9)	$\therefore r$	(M.P. 7 and 8)

4. [3] Write the statement

$$\neg[\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, (x = 0) \to (xy \neq 1)]$$

in symbols without using any negated quantifiers or negated mathematical symbols and determine its truth value.

$$\neg[\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, (x=0) \to (xy \neq 1)] \overset{①}{\Leftrightarrow} \forall x \in \mathbb{Z}, \neg[\forall y \in \mathbb{Z}, (x=0) \to (xy \neq 1)]$$

$$\overset{②}{\Leftrightarrow} \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \neg[(x=0) \to (xy \neq 1)]$$

$$\overset{③}{\Leftrightarrow} \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \neg(\neg(x=0) \vee (xy \neq 1))$$

$$\overset{④}{\Leftrightarrow} \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, (x=0) \wedge \neg(xy \neq 1)$$

$$\overset{⑤}{\Leftrightarrow} \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, (x=0) \wedge (xy = 1).$$

① By Negation of $\exists$
② Negation of $\forall$
③ As $p \to q \Leftrightarrow \neg p \vee q$
④ De Morgan's
⑤ Negation of $xy \neq 1$ is $xy = 1$.

The statement is false because for every ~~x~~ x in $\mathbb{Z}$, either $x \neq 0$ and so the statement is false, or $x = 0$, but then $xy \neq 1$, so the statement is false.

5. Let k be an integer.

(a) [2] Prove that if k is even, then k^2 is even.

Suppose that k is even. Then $\exists t \in \mathbb{Z}$ such that $k = 2t$.
But now, $k^2 = (2t)^2 = 2 \cdot (2t^2)$. So k^2 is even.

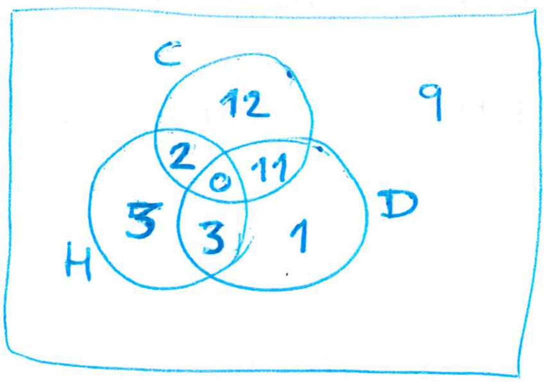
(b) [1] Write the converse of the statement in (a). Is the converse logically equivalent to the statement in (a)?

If k^2 is even, then k is even.

They are logically equivalent because they are both tautologies.

6. [3] There are 41 MATH 122 students. 25 own cats. 15 own dogs. 8 own hamsters. No student has a cat and a dog and a hamster. 11 have a cat and a dog. 2 have a cat and a hamster. Exactly one student with a dog does not have another pet. Draw a Venn diagram to represent the situation and determine how many students have no pets.

let C be the set of students with cats.
let D be the set of students with dogs
let H be the set of students with hamsters.



9 students have no pets.

7. [3] Let A and B be sets such that $A \cup B = B$. Prove that $A \cap B = A$. Is the converse true?

Suppose that $A \cup B = B$. We will show

① $A \subseteq A \cap B$ and ② $A \cap B \subseteq A$. This suffices to show $A \cap B = A$.

For ①, take $x \in A$. As $A \cup B = B$, $x \in B$. So $x \in A \wedge x \in B$ and so $x \in A \cap B$. So $A \subseteq A \cap B$.

For ②, by definition of intersection, $A \cap B \subseteq A$.

As we have shown $LHS \subseteq RHS$ and $RHS \subseteq LHS$, we have proved that $A \cap B = A$.

The converse is $(A \cap B = A) \to (A \cup B = B)$.

~~this~~ This is true. If $A \cap B = A$, then $A \subseteq A \cap B$ and so $A \subseteq B$. Hence $A \cup B \subseteq B$. As $B \subseteq A \cup B$ (by definition), this completes the proof.

So the converse is not a contradiction (as it is always true).