1. [6] Use the **bubble sheet** provided on the last page of the test booklet to indicate whether each statement is **True (A)** or **False (B)**.

T [1] $1 + 5 + 5^2 + \cdots + 5^n = \frac{1}{4}(5^{n+1} - 1)$.

F [2] When using induction to prove a statement $S(n)$ for all $n \geq 0$, in the induction hypothesis you assume $S(n)$ is true for all n.

F [3] Let $a_0, a_1, \ldots$ be the sequence recursively defined by $a_0 = 1$, $a_1 = 3$ and $a_n = 2a_{n-1} - a_{n-2}$ for all $n \geq 2$. Then $a_4 = 7$. $a_2 = 2\times3-1=5$ $a_3 = 2a_2 - a_1 = 7$ $a_4 = 2\times7-5 = 9$

F [4] If a statement $S(n)$ is true for every $n \in \{1, 2, \ldots, k\}$, then $S(n)$ is true when $n = k + 1$.

F [5] Let $a, b \in \mathbb{Z}$. If $\gcd(a, b) = 1$, then there are no integers k, ℓ such that $ka + \ell b = 1$.

T [6] $36^{35} - 343^{123} \equiv 1$ (modulo 7) $36 \equiv 1\ (7)$, $343 \equiv 0\ (7)$

F [7] $(221)_3 = 17$. $(221)_3 = 1\times3^0 + 2\times3 + 2\times3^2 = 25$

F [8] There are integers a and b such that $2 \cdot 3^a = 6 \cdot 5^b$.

F [9] Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Then $(b, 2) \in A \times B$.

F [10] $A \times B = B \times A$ only when $A = B$.

F [11] Let $A = \{1, 2, 3, 4\}$. The relation on A defined by $\mathcal{R} = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$ is reflexive. $(4,4) \notin R$

T [12] Let $\mathcal{R}$ be the relation on $\mathbb{N}$ defined by $a \sim b$ in $\mathcal{R}$ if and only if $\frac{a}{b} \leq \frac{b}{a}$. $\mathcal{R}$ is antisymmetric.

If $a \sim b$ and $b \sim a$

then $\frac{a}{b} \leq \frac{b}{a}$ and $\frac{b}{a} \leq \frac{a}{b}$

So $a^2 \leq b^2 \leq a^2$

$\Rightarrow a = b$.

2. [5] Use induction to prove that $3 + 6 + 9 + \cdots + 3n = \frac{3n(n+1)}{2}$, for all $n \geq 1$.

Base: When $n=1$, $3n = 3 = \frac{3 \times 1(1+1)}{2}$

So the statement holds when $n=1$.

IH: Suppose $3+6+9+\cdots+3n = \frac{3n(n+1)}{2}$

for all $n \in \{1,2,\ldots,k\}$.

IS: We want to show that $3+6+9+\cdots+3n+3(n+1) = \frac{3(n+1)(n+2)}{2}$.

$$3+6+9+\cdots+3n+3(n+1) = \frac{3n(n+1)}{2} + 3(n+1), \text{ (by IH}$$

$$= \frac{3n(n+1)+6(n+1)}{2}$$

$$= \frac{(3n+6)(n+1)}{2}$$

$$= \frac{3(n+2)(n+1)}{2},$$

as required.

Conclusion: So by induction, $3+6+9+\cdots+3n = \frac{3n(n+1)}{2}$

for all $n \geq 1$.

4. (a) [2] Use the Euclidean algorithm to find gcd(1234, 567).

$$1234 = 567 \times 2 + 100$$

$$567 = 100 \times 5 + 67$$

$$100 = 67 \times 1 + 33$$

$$67 = 33 \times 2 + 1$$

$$33 = 33 \times 1 + 0$$

$$\gcd(1234, 567) = \gcd(567, 100) = \gcd(100, 67) = \gcd(67, 33)$$
$$= \gcd(33, 1) = 1$$

So $\gcd(1234, 567) = 1$

(b) [1] Use your solution from part (a) to find lcm(1234, 567).

$$\gcd(1234, 567) \times \operatorname{lcm}(1234, 567) = 1234 \times 567$$

So $\operatorname{lcm}(1234, 567) = 1234 \times 567$

3. [3] Let $a, b, d \in \mathbb{Z}$. Prove that if $d|b$, then $ad|ab^2$.

4. Suppose $d|b$. Then there exists some integer $k \in \mathbb{Z}$ such that $b = kd$. But now, $b^2 = (kd)^2 = k^2d^2$ and so $ab^2 = ak^2d^2 = (k^2d)ad$

So as $k, d \in \mathbb{Z}$, $(k^2d) \in \mathbb{Z}$ and so setting $l = k$ we see there is an integer $l \in \mathbb{Z}$ such that

$ab^2 = lad$. So $ad|ab^2$.

5. [4] Let $\mathcal{R}$ be the relation on $\mathbb{Z}$ defined by $a \sim b$ if and only if $3|a - b$. Prove that $\mathcal{R}$ is an equivalence relation.

We must show that $\mathcal{R}$ is reflexive, symmetric and transitive.

Reflexive: let $a \in \mathbb{Z}$. As $a - a = 0$ and $3|0$, ~~th~~ $3|a-a$ so $a \sim a$ and $\mathcal{R}$ is reflexive.

Symmetric: let $a, b \in \mathbb{Z}$. Suppose $a \sim b$, that is $3|a-b$. Then $\exists k \in \mathbb{Z}$ such that $3k = a - b$. But then $-3k = b - a$ and so $3(-k) = b - a$ which means $3|b-a$. So $b \sim a$ and $\mathcal{R}$ is symmetric.

Transitive: let $a, b, c \in \mathbb{Z}$. Suppose $a \sim b$ and $b \sim c$. Then $3|a-b$ and $3|b-c$.
So $\exists k \in \mathbb{Z}$ such that $3k = a - b$ (1) and $\exists \ell \in \mathbb{Z}$ such that $3\ell = b - c$ (2).

Summing (1) and (2) gives $3(k+\ell) = a - c$

So $3|a-c$ and $a \sim c$ so $\mathcal{R}$ is transitive.

As $\mathcal{R}$ is reflexive, symmetric and transitive, it is an equivalence relation.

6. [1] Let $a, d \in \mathbb{R}$. Write a recursive definition for the sequence $a, a+d, a+2d, a+3d, \ldots$.

Base case : $a_1 = a$

Recursion: $a_n = a_{n-1} + d$, for all $n \geqslant 2$.

7. [2] Let A, B, C be sets. Prove that $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

Take any $(x,y) \in (A \times B) \cap (A \times C)$. Then $(x,y) \in A \times B$ and $(x,y) \in A \times$

As $(x,y) \in A \times B$, by defn of $A \times B$, $x \in A$ and $y \in B$.

Similarly, as $(x,y) \in A \times C$, $x \in A$ and $y \in C$.

So $x \in A$ and $y \in B \cap C$, which implies $(x,y) \in A \times (B \cap C)$.

~~So~~ As (x,y) was arbitrary, $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.