202401 Math 122 A01 Quiz #3

#V_________________ Name: Key

The time limit is 25 minutes. There are total of 15 marks available on the two pages. You may use a Sharp calculator with model number beginning EL510R. Except where indicated, you must show your work in order to receive full or partial credit.

1. [2] Use the blank to indicate whether each statement is true (T) or false (F). No reasons are necessary.

   F If $3, 8 \in X$ and $x + y \in X$ whenever $x, y \in X$, then $13 \in X$.

   F If $a_1 = 0,\ a_2 = 1$ and $a_n = a_{n-2}+1$ for $n \geq 3$, then $a_1, a_2, \ldots = 0, 1, 0, 2, 0, 3, 0 \ldots$.

   F If $S(n)$ is an open statement in the universe of positive integers and the truth of $S(k)$ logically implies the truth of $S(k+1)$ for all $k \geq 1$, then $S(n)$ is true for all positive integers n.

   T $1^3 + 2^3 + \cdots + n^3 = n^2(n+1)^2/4$ for all $n \geq 0$.

2. [4] Let $a_0 = 1,\ a_1 = 2$, and $a_n = 4a_{n-1} - 4a_{n-2},\ n \geq 2$. Use induction to prove that $a_n = 2^n$ for all integers $n \geq 0$.

Basis: When $n=0$ we have $a_0 = 1 = 2^0$ ✓
when $n=1$ we have $a_1 = 2 = 2^1$ ✓

$\therefore$ The statement is true when $n=0$ and when $n=1$.

IH: Suppose there is an integer $k \geq 1$ such that $a_n = 2^n$ for $n = 0, 1, \ldots, k$

IS: We need to show that $a_{k+1} = 2^{k+1}$.

Look at a_{k+1}. Since $k+1 \geq 2$ we can use the recurrence:

$$\begin{aligned} a_{k+1} &= 4a_k - 4a_{k-1} \\ &= 4 \cdot 2^k - 4 \cdot 2^{k-1} \\ &= 2^2 \cdot 2^k - 2^2 2^{k-1} \\ &= 2 \cdot 2^{k+1} - 2^{k+1} \\ &= 2^{k+1}, \text{ as wanted} \end{aligned}$$

$\therefore$ By induction, $a_n = 2^n \ \forall n \geq 0$

3. [2] Give a recursive definition of the sequence $s_0, s_1, s_2, \ldots = -2, 4, 10, 16, \ldots, 6n-2, \ldots$.

$S_0 = -2$

$S_n = S_{n-1} + 6 \quad , n \geq 1$

4. [2] Gary is proving by induction the (true) statement that *every integer* $n \geq 18$ *can be written as a sum of 4s and 7s.* He first verifies the truth of the statement for $n = 18, 19, 20$ and 21. Now he needs to state the induction hypothesis. Use the space below to help him out by telling him what to write.

Suppose there is an integer $k \geq 21$ such that every integer n with $18 \leq n \leq k$ can be written as a sum of 4s and 7s

5. Let $s_0 = 7$, and $s_n = 2s_{n-1} + 7$ for $n \geq 1$.

(a) [1] Use the recursive definition to find expressions for s_1, s_2, s_3, s_4 as a sum of powers multiplied by constants.

$S_1 = 2S_0 + 7 = 2 \cdot 7 + 7$

$S_2 = 2S_1 + 7 = 2(2 \cdot 7 + 7) + 7 = 2^2 \cdot 7 + 2 \cdot 7 + 7$

$S_3 = 2S_2 + 7 = 2(2^2 \cdot 7 + 2 \cdot 7 + 7) + 7$

$= 2^3 \cdot 7 + 2^2 \cdot 7 + 2 \cdot 7 + 7$

$S_4 = 2S_3 + 7 = 2(2^3 \cdot 7 + 2^2 \cdot 7 + 2 \cdot 7 + 7) + 7$

$= 2^4 \cdot 7 + 2^3 \cdot 7 + 2^2 \cdot 7 + 2 \cdot 7 + 7$

(b) [2] Use your work in part (a) to guess a closed-form formula (i.e. not just a summation) for s_n that is valid for all $n \geq 0$. You do not need to prove that your formula is correct.

Guess : $S_n = 2^n \cdot 7 + 2^{n-1} \cdot 7 + \cdots + 2 \cdot 7 + 7$

$= 7(1 + 2 + \cdots + 2^n)$

$= 7 \dfrac{(2^{n+1} - 1)}{2-1} = 7(2^{n+1} - 1)$

6. [2] Use the blank to indicate whether each statement is true (T) or false (F). No reasons are necessary.

T If $x \in \mathbb{R} \setminus \mathbb{Z}$ then $\lfloor x \rfloor < \lceil x \rceil$.

F If $x \in \mathbb{R}$, then $\lfloor 2x \rfloor < 2\lfloor x \rfloor$.

F When -41 is divided by 5, the remainder is -1.

F The remainder when $(2503)_9$ is divided by 9 equals 2.