201701 Math 122 Assignment 3 Sol'n Ideas

1. ($a \Rightarrow b$) Suppose $A \subseteq B$. $\therefore$ If $x \in A$ then $x \in B$.
The contrapositive is if $x \notin B$ then $x \notin A$, i.e., $B^c \subseteq A^c$

($b \Rightarrow c$) Suppose $B^c \subseteq A^c$. We want to show $A \cap B \subseteq A$ and $A \subseteq A \cap B$. The first of these is true by def'n of intersection. We prove the second.

Suppose $x \in A$. Then $x \notin A^c$. Since $B^c \subseteq A^c$, $x \notin B^c$.
$\therefore$ $x \in B$. $\therefore$ $x \in A \cap B$. $\therefore$ $A \subseteq A \cap B$
Since $A \subseteq A \cap B$ and $A \cap B \subseteq A$, it follows that $A = A \cap B$.

($c \Rightarrow d$) Suppose $A \cap B = A$. We show $A \setminus B = \emptyset$ by contradiction.

Suppose $A \setminus B \neq \emptyset$ and let $x \in A \setminus B$. Then $x \in A$ & $x \notin B$.

$\therefore x \notin A \cap B$, so $A \cap B \neq A$, a contradiction.

$\therefore A \setminus B = \emptyset$.

$(d \Rightarrow a)$ Suppose $A \setminus B = \emptyset$. Take any $x \in A$.

Then $x \notin A \setminus B$. $\therefore x \in B$. $\therefore A \subseteq B$.

2. a) Take any $(x,y) \in A \times C$. Then $x \in A$ and $y \in C$.

Since $A \subseteq B$, $x \in B$. Since $C \subseteq D$, $y \in D$.

$\therefore (x,y) \in B \times D$. $\therefore A \times C \subseteq B \times D$.

We need to show it is a proper subset.

Since $A \subsetneq B$, there exists $b \in B$ s.t. $b \notin A$.

Since $D \neq \emptyset$, there exists $d \in D$.

Then $(b,d) \in B \times D$ but $(b,d) \notin A \times C$ (as $b \notin A$).

$\therefore A \times C \subsetneq B \times D$.

b) Yes. We show $(x,y) \in A \times (B \setminus C)$
$\iff (x,y) \in (A \times B) \setminus (A \times C)$.

We have $(x,y) \in A \times (B \setminus C)$

$$\begin{aligned}
&\iff (x \in A) \wedge (y \in B \setminus C) \\
&\iff (x \in A) \wedge (y \in B) \wedge (y \notin C) \\
&\iff [(x,y) \in A \times B] \wedge [(x,y) \notin A \times C] \\
&\iff (x,y) \in (A \times B) \setminus (A \times C)
\end{aligned}$$

3. a) False. $R = \{(1,2)\}$ & $S = \{(2,1)\}$ are anti-symmetric, but $R \cup S = \{(1,2),(2,1)\}$ isn't b/c $(1,2),(2,1) \in R \cup S$ and $1 \neq 2$.

b) False. $R = \emptyset$ is symmetric & transitive but not reflexive (on $A = \{1,2,3\}$) b/c, for example, $(1,1) \notin R$.

c) True. Suppose $(a,b),(b,c) \in R \cap S$. Then $(a,b),(b,c) \in R$ and $(a,b),(b,c) \in S$. Since R & S are transitive, $(a,c) \in R$ and $(a,c) \in S$. $\therefore (a,c) \in R \cap S$
$\therefore R \cap S$ is transitive.

4. <u>reflexive</u>. Yes. For any $X \subseteq S$, $X \oplus X = \emptyset$
$\therefore (X,X) \in R$ $\therefore$ R is reflexive.

<u>symmetric</u>. Yes. Suppose $X, Y \subseteq S$ and $(X,Y) \in R$.
Then $X \oplus Y = \emptyset$. Since $X \oplus Y = Y \oplus X$
by def'n, $Y \oplus X = \emptyset$. $\therefore (Y,X) \in R$
$\therefore$ R is symmetric.

<u>transitive</u>. Yes. Suppose $X, Y, Z \subseteq S$ and
$(X,Y), (Y,Z) \in R$. $\therefore X \oplus Y = \emptyset$ and $Y \oplus Z = \emptyset$
We show $X \oplus Z = \emptyset$ by contradiction.
Suppose $X \oplus Z \neq \emptyset$ and let $w \in X \oplus Z$.
Then $w \in X \setminus Z$ or $w \in Z \setminus X$.

<u>Case 1</u> $w \in X \setminus Z$.
Then $w \in X$ and $w \notin Z$.
If $w \in Y$, then $w \in Y \oplus Z$, so $Y \oplus Z \neq \emptyset \Rightarrow\Leftarrow$
If $w \notin Y$, then $w \in X \oplus Y$, so $X \oplus Y \neq \emptyset \Rightarrow\Leftarrow$

<u>Case 2</u> $w \in Z \setminus X$.
Then $w \in Z$ and $w \notin X$.
If $w \in Y$, then $w \in X \oplus Y$, so $X \oplus Y \neq \emptyset \Rightarrow\Leftarrow$
If $w \notin Y$, then $w \in Y \oplus Z$, so $Y \oplus Z \neq \emptyset \Rightarrow\Leftarrow$

Both cases give a contradiction

$\therefore X \oplus Z = \phi$, and R is transitive

<u>anti-symmetric</u>. Yes. Suppose $X, Y \subseteq S$ with $(X,Y), (Y,X) \in R$. Then $X \oplus Y = \phi$ and $Y \oplus X = \phi$.
We need to show that $X = Y$, so we prove $X \subseteq Y$ & $Y \subseteq X$.
Take any $x \in X$. Since $x \notin X \oplus Y$, $x \in Y$ $\therefore X \subseteq Y$.
Take any $y \in Y$. Since $y \notin Y \oplus X$, $y \in X$ $\therefore Y \subseteq X$.
$\therefore X = Y$ $\therefore R$ is anti-symmetric.

<u>Note</u>. All of the above could have been done differently by 1st showing $X \oplus Y = \phi \Leftrightarrow X = Y$ (as we just did!)

5. a) reflexive. Let $s \in S$. The entries of s can be rearranged to give the same sequence: do nothing! $\therefore (s,s) \in R$ and R is reflexive.

symmetric. Let $s_1, s_2 \in S$ s.t. $(s_1, s_2) \in R$. Then the entries of s_1 can be rearranged to give s_2. The reverse sequence of moves rearranges the entries of s_2 to give s_1. $\therefore (s_2, s_1) \in R$ and R is symmetric

<u>transitive</u>. Let $s_1, s_2, s_3 \in S$ s.t. $(s_1, s_2), (s_2, s_3) \in R$
Then the entries of s_1 can be rearranged to give s_2, &
" " " s_2 " " " " " s_3.

Since making the 1^{st} sequence of moves and then the second one transforms s_1 into s_3, we have $(s_1, s_3) \in R$ and R is transitive.

not anti-symmetric. We have $(1000, 0100) \in R$ and $(0100, 1000) \in R$ but $1000 \neq 0100$

$\therefore$ R is not anti-symmetric.

b) $[0000] = \{0000\}$

$[1000] = \{1000, 0100, 0010, 0001\} = [0100] = [0010] = [0001]$

$[1100] = \{1100, 1010, 1001, 0110, 0101, 0011\}$

$= [1010] = [1001] = [0110] = [0101] = [0011]$

$[1110] = \{1110, 1101, 1011, 0111\} = [1101] = [1011] = [0111]$

$[1111] = \{1111\}$

i) If $(x, y) \in R$, then $[x] = [y]$.

ii) If $(x, y) \notin R$, then $[x] \cap [y] = \phi$.