202409 Quiz 4
Math 122 A04
Instructor: Jonathan Noel

Answer Key

Do not open the booklet before you are told to

Date and Time: Tuesday, October 29, 2024 at 1:55pm.

Instructions: There are 3 pages and 4 questions. There are 15 marks available. The time limit is 25 minutes. Math and Stats standard calculators are allowed. Except when indicated, it is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the sheet for rough work.

True/False Instructions: Question 1 consists of 8 true/false questions labelled **TF 1** to **TF 8**. The last page of your test booklet is a bubble sheet for answering them. You may detach the back page from the rest of the test if you wish. Only fill in a bubble for questions 1-8 on the bubble sheet. When making your selection, **True is A** and **False is B**. Do not select C, D or E. If you want to change your answer after filling in a bubble, then please erase your previous answer or write something on the sheet to try to make your final selection as clear as possible.

1. [4] Use the bubble sheet provided on the last page of the test booklet to indicate whether each statement is **True (A)** or **False (B)**.

Let $A = \{1, 2, \ldots 8\}$.

[TF 1] The number of proper subsets of A containing 2 is $2^7 - 1$.

This is **true**. The number of subsets containing 2 is 2^7, since there is only one choice for 2 and there are two choices for all other elements. The only non-proper subset containing 2 is A itself, and so there are $2^7 - 1$ proper subsets containing 2.

[TF 2] The number of non-empty subsets of A containing 2 is $2^7 - 1$.

This is false. There are 2^7 subsets containing 2 and all of them are non-empty.

[TF 3] $\{5\}$ is a subset of the power set of A.

This is false. The set $\{5\}$ is an element of the power set of A, but it is not a subset of it. This is because 5 is an element of $\{5\}$ but it is not an element of $\mathcal{P}(A)$.

[TF 4] If $B \subseteq A$, then $|A \setminus B| < 8$.

This is false. Consider $B = \emptyset$.

[TF 5] $4^0 + 4^1 + \cdots + 4^n = 5^n$ for all integers $n \geq 0$.

This is false. For $n = 2$, the LHS is $4^0 + 4^1 + 4^2 = 1 + 4 + 16 = 21$ and the RHS is $5^2 = 25$.

[TF 6] If $x, y \in \mathbb{Q}$ such that $\lfloor x \rfloor < \lceil y \rceil$, then $x < y$.

This is false. Consider $x = 0.6$ and $y = 0.5$. Then $\lfloor x \rfloor < \lceil y \rceil$ but $x > y$.

[TF 7] Let $a_0, a_1, \ldots$ be the sequence recursively defined by $a_0 = 0$, $a_1 = 3$ and $a_n = 3a_{n-1} - a_{n-2}$ for all $n \geq 2$. Then $a_3 = 27$.

This is false. We have $a_2 = 3 \cdot 3 - 0 = 9$ and $a_3 = 3 \cdot 9 - 3 = 24$.

[TF 8] The base 8 representation of 3000 is $(5607)_8$.

This is false. We have $(5607)_8 = 5 \cdot 8^3 + 6 \cdot 8^2 + 0 \cdot 8^1 + 7 \cdot 8^0 = 2951$. Note that, since $(5607)_8$ endes with a 7, it will be odd and so you can tell that it is not equal to 3000 without even computing it.

2. [1] Give a recursive definition of the sequence $a_1, a_2, \ldots$, where $a_n = 1^1 + 2^2 + 3^3 + \cdots + n^n$, for every $n \geq 1$.

$a_1 = 1^1$ and $a_n = a_{n-1} + n^n$ for $n \geq 2$.

3. [5] Let $a_0, a_1, \ldots$ be the sequence recursively defined by $a_0 = 0$, $a_1 = 2$ and $a_n = 2a_{n-1} - a_{n-2}$ for all $n \geq 2$. Prove by induction that $a_n = 2n$ for all integers $n \geq 0$.

Basis: Consider $n = 0$. We have LHS $= a_0 = 0$ and RHS $= 2 \cdot 0 = 0$. Also, consider $n = 1$. We have LHS $= a_1 = 2$ and RHS $= 2 \cdot 1 = 2$. So, LHS $=$ RHS in both cases and so the statement is true for $n = 0$ and $n = 1$.

Common Mistakes: Many people only proved the case $n = 0$ in the basis. However, it is important to do $n = 1$ as well because the recursive definition $a_n = 2a_{n-1} - a_{n-2}$ cannot be used for $n = 1$ (it requires $n \geq 2$). So, if you do not prove the case $n = 1$ in the basis, it will be impossible to prove it in the IS.

Another common mistake was to start the basis at $n = 2$. However, this is not sufficient because you were asked to prove that $a_n = 2n$ for all $n \geq 0$. If you do not prove it for $n = 0$ and $n = 1$, then you have not proved it for all $n \geq 0$.

Inductive Hypothesis: Let $n \geq 1$ and assume that $a_k = 2k$ for all k such that $0 \leq k \leq n$.

Common Mistakes: Many people only assumed one case in the IH. For example, they wrote "let $n \geq 1$ and assume that $a_n = 2n$." However, this cannot work because, in order to prove the statement $a_{n+1} = 2(n + 1)$, you need to use that $a_n = 2n$ and $a_{n-1} = 2(n - 1)$. Just knowing one of these statements is not going to be good enough.

Another mistake that many people made was to write something like "Assume $a_n = 2n$ for all $n \geq 0$" as their IH. This is a really big problem because it is assuming the full statement that you want to prove!

Yet another mistake was getting k and n mixed up. For example, many people wrote things like "Let $n \geq 1$ and assume that $a_k = 2k$ for $n = 0, 1, \ldots, k$." This statement does not make sense. To be clear, there is nothing wrong with the statement "Let $k \geq 1$ and assume that $a_n = 2n$ for all $n = 0, 1, \ldots, k$." This would get full marks even though the n and the k are playing different "roles" than in the solution I wrote above. However, if you wrote it like this, then your IS should be in terms of a_{k+1} not a_{n+1}.

Another mistake that people made was to not declare their variables. A lot of people wrote things like "Assume that $a_n = 2n$ for $n = 0, 1, \ldots, k$." However, in this statement, k is never introduced. So, where did k come from? Until you declare what k is (by saying something like "let $k \geq 1$"), you cannot refer to it.

Lastly, another common mistake was to write the IH as something like "Let $n \geq 2$ and assume that $a_n = 2a_{n-1} - a_{n-2}$." But this is not the statement that you are trying to prove!

Comment: Some people wrote their IH as something like this: "Let $n \geq 1$ and assume that $a_k = 2k$ for $k = n$ and $k = n - 1$." This is perfectly acceptable. In fact, this shows that you understand *exactly* what you need to assume in the IH to prove the IS, which is great! However, unfortunately, a lot of people who tried to do this also mixed up their variables n and k or forgot to declare them. But, if you avoided those mistakes, then the idea would have been good.

Inductive Step: We prove that $a_{n+1} = 2(n+1)$. We have

$$\begin{aligned} a_{n+1} &= 2a_n - a_{n-1} \quad \text{(by the recursive definition, since } n+1 \geq 2) \\ &= 2 \cdot 2n - 2(n-1) \quad \text{(by IH)} \\ &= 4n - 2n + 2 \\ &= 2(n+1). \end{aligned}$$

Common Mistakes: The most common mistake in the IS was to forget to verify that the recursive definition can be applied. The recursive definiton $a_n = 2a_{n-1} - a_{n-2}$ can only be applied when the "index" n is at least two. So, in order to apply it, you need to reference the fact that the index is at least two. Some people even tried to apply $a_n = 2a_{n-1} - a_{n-2}$ in the case $n = 1$, which is not possible because, if $n = 1$, then $a_{n-2} = a_{-1}$ which is undefined.

The second most common mistake was to forget to reference the IH. Whenever you do a proof by induction, there should *always* be a reference to the IH somewhere in the IS!

There were also some cases where students just computed a few terms (like a_2, a_3, a_4) and seemed to think that this was enough to prove the statement. However, this is not sufficient to prove the statement for all n.

Quite a lot of people also seemed to think that they needed to prove that $a_{k+1} = 2k + 1$. However, it should have been $a_{k+1} = 2(k+1)$.

Conclusion: $a_n = 2n$ for all $n \geq 0$.

Common Mistakes: The conclusion should always be "free marks" for just knowing what you have proven and writing it down! A lot of people wrote something like $a_n = 2a_{n-1} - a_{n-2}$ for their conclusion, which is not correct because this is not what you were asked to prove.

Many people wrote an unnecessarily long conclusion like "Since the basis is true for $n = 0$ and $n = 1$ and the inductive hypothesis implies the statement for $n+1$, by PMI, the statement $a_n = 2n$ is true when a_n is the sequence defined by $a_0 = 0$, $a_1 = 2$ and $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$." But there is really no reason to write all of this! If you did write a long statement and it was all correct, then that is fine. However, many students wrote very long conclusions that included a few incorrect or imprecise statements and lost marks for that. If they just wrote a short conclusion like the one I wrote above to show that they understand what they proved, then they would not have lost those marks!

4. [5] Use induction to prove that $1+4+7+\cdots+(3n-2)=\frac{n(3n-1)}{2}$ for all integers $n \geq 1$.

Basis: Consider the case $n=1$. Then LHS $=1$ and RHS $=\frac{1(3\cdot 1-1)}{2}=1$. So, LHS $=$ RHS in the case $n=1$.

Comment: Some of you may be wondering: "why are we only doing one basis case for this proof, but we did two for the previous question?" The answer is that we only need one for this question, but we needed two for the previous one. If you are trying to just memorize how to write an induction proof without understanding *why* induction works, then it will probably be very unclear to you why this is the case! I would really recommend to avoid just trying to memorize the steps. It is much better to deeply understand why induction works. If you understand it, then it will be completely clear why we are only proving one basis case here. The reason is that we only need to go "one step back" when we do our induction step. But, in the previous question, we needed to go "two steps back."

Inductive Hypothesis: Let $n \geq 1$ and assume that $1+4+7+\cdots+(3n-2)=\frac{n(3n-1)}{2}$.

Comment: You may be wondering: How come the IH in the previous question had a k in it and this one doesn't? Again, if you're just trying to memorize the pattern of induction proofs without understanding how they work, then this will seem weird! However, I really encourage you to understand deeply how induction works and what "role" the IH plays in an induction proof, and then it will all be clear!

Induction Step: We show that $1+4+7+\cdots+(3n-2)+(3(n+1)-2)=\frac{(n+1)(3(n+1)-1)}{2}$. We have

$$
\begin{aligned}
&1+4+7+\cdots+(3n-2)+(3(n+1)-2)\\
&=(1+4+7+\cdots+(3n-2))+(3(n+1)-2)\\
&=\frac{n(3n-1)}{2}+(3(n+1)-2) \quad \text{(by IH)}\\
&=\frac{n(3n-1)+2(3(n+1)-2)}{2}\\
&=\frac{n(3n-1)+6n+2}{2}\\
&=\frac{n(3n-1)+(3n-1)+(3n+3)}{2}\\
&=\frac{(n+1)(3n-1)+(3n+3)}{2}\\
&=\frac{(n+1)(3n+2)}{2}\\
&=\frac{(n+1)(3(n+1)-1)}{2}.
\end{aligned}
$$

Conclusion: $1+4+7+\cdots+(3n-2)=\frac{n(3n-1)}{2}$ for all $n \geq 1$.