# 201709 Math 122 A01 Quiz #4

#V00:________________ Name: Key

This quiz has 2 pages and 4 questions. There are 15 marks available. The time limit is 25 minutes. Math and Stats standard calculators are allowed, but no calculator is needed. It is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the pages for rough or extra work.

1. Suppose you know that there is an integer $k \geq 3$ such that $n^2 < 3^n$ for all integers $n$ that satisfy $0 \leq n \leq k$.

   (a) [3] Show that $(k+1)^2 < 3^{k+1}$.

   We have $(k+1)^2 = k^2 + 2k + 1$

   $< k^2 + k \cdot k + k \cdot k$ since $k \geq 3$

   $= 3 \cdot k^2$

   $< 3 \cdot 3^k$ by hypothesis

   $= 3^{k+1}$, as needed.

   (b) [1] Is $n^2 < 3^n$ for all integers $n \geq 0$? Explain.

   Yes. Taken as a whole, part (a) is a proof by induction of this statement.

2. [2] Give a recursive definition the sequence

$$s_0, s_1, \ldots, s_n, \ldots = -1, 5, 11, \ldots, 6n-1, \ldots$$

$s_0 = -1$

$s_n = s_{n-1} + 6, \quad n \geq 1.$

3. Let $a_0, a_1, a_2, \ldots$ be the sequence recursively defined by $a_0 = 1$, and $a_n = 5a_{n-1} + 1$.

(a) [2] Find $a_1, a_2, a_3$, and $a_4$. Leave your answer as a sum rather than computing a numerical value.

$a_1 = 5a_0 + 1 = 5+1$
$a_2 = 5a_1 + 1 = 5(5+1)+1 = 5^2+5+1$
$a_3 = 5a_2 + 1 = 5(5^2+5+1)+1 = 5^3+5^2+5+1$
$a_4 = 5a_3 + 1 = 5(5^3+5^2+5+1) = 5^4+5^3+5^2+5+1$

(b) [2] Based on your work in part (a), conjecture a formula for $a_n, n \geq 0$. (Note: a formula, not a summation).

$a_n = 5^n + 5^{n-1} + \cdots + 5 + 1$
$= (5^{n+1}-1)/(5-1) = \frac{5^{n+1}-1}{4}$

4. [5] Let $b_n$ be the sequence recursively defined by $b_0 = 1, b_1 = 3$ and $b_n = 6b_{n-1} - 9b_{n-2}$. Prove by induction that $b_n = 3^n$, for all $n \geq 0$.

Basis: When $n=0$ we have $b_0 = 1 = 3^0$ ✓
When $n=1$ we have $b_1 = 3 = 3^1$ ✓
∴ The statement is true when $n=0$ & when $n=1$.

IH: For some $k \geq 1$ assume:
$b_0 = 3^0, b_1 = 3^1, \ldots, b_k = 3^k$.

IS: Want: $b_{k+1} = 3^{k+1}$
Look at $b_{k+1}$. Since $k+1 \geq 2$, we have
$b_{k+1} = 6b_k - 9b_{k-1}$
$= 6 \cdot 3^k - 9 \cdot 3^{k-1}$
$= 2 \cdot 3^{k+1} - 3^{k+1}$ $(6 = 2 \cdot 3$ & $9 = 3^2)$
$= 3^{k+1}$, as wanted

∴ By induction, $b_n = 3^n$ $\forall n \geq 0$