201809 Math 122 A01 Quiz #4

#V00:______________ Name: Solutions

This quiz has 2 pages and 4 questions. There are 15 marks available. The time limit is 25 minutes. Math and Stats standard calculators are allowed, but not needed. Except when indicated, it is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the pages for rough or extra work.

1. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is needed. Let $A = \{1, 2, \ldots, 13\}$.

   F The number of proper subsets of A that do not contain 13 equals $2^{12} - 2$.

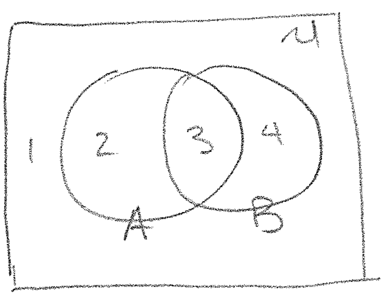
   T The number of subsets of A that contain 1, 2 and 3 equals 2^{10}.

   T The number of subsets of A that contain 1, or neither 2 nor 3, equals $2^{12}+2^{11}-2^{10}$.

   F If B is a set of integers such that $|B| = 20$ and $|A \cup B| = 25$, then $|A \setminus B| = 7$.

2. Let A and B be sets.

   (a) [2] Give a counterexample to show that the statement $(A \oplus B)^c = A \cap B$ for all sets A, B is false.



Let $\mathcal{U} = \{1,2,3,4\}$, $A = \{2,3\}$, $B = \{3,4\}$.
Then $A \cap B = \{3\}$ whereas $(A \oplus B)^c = \{2,4\}^c = \{1,3\}$.
$\therefore$ The statement $(A \oplus B)^c = A \cap B \;\forall A, B$ is false.

   (b) [2] Use the Laws of Set Theory to show that $(A \cup B) \cap (B \setminus A)^c = A$.

$(A \cup B) \cap (B \setminus A)^c$

$= (A \cup B) \cap (B \cap A^c)^c$ Known

$= (A \cup B) \cap (A \cup B^c)$ DeMorgan + Comm

$= A \cup \underbrace{(B \cap B^c)}_{\emptyset}$ Dist

$= A \cup \emptyset = A$ Known.

3. [5] Use induction to prove that $1 + 3 + \cdots + (2n-1) = n^2$, for all integers $n \geq 1$.

This appears in the notes,

Section 4.4.1, page 11

4. Let $a_0, a_1, \ldots$ be the sequence defined by $a_0 = 3$ and $a_n = 2a_{n-1} + 3$ for $n \geq 1$.

(a) [2] Express a_1, a_2, a_3, and a_4 as summations.

$$a_1 = 2a_0 + 3 = 2 \cdot 3 + 3$$
$$a_2 = 2a_1 + 3 = 2(2 \cdot 3 + 3) + 3$$
$$= 2^2 \cdot 3 + 2 \cdot 3 + 3$$
$$a_3 = 2a_2 + 3 = 2(2^2 \cdot 3 + 2 \cdot 3 + 3)$$
$$= 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3$$
$$a_4 = 2a_3 + 3 = 2(2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 2)$$
$$= 2^4 \cdot 3 + 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3$$

(b) [2] Use your work in part (a) to conjecture (i.e., guess) a formula for a_n which is valid for all $n \geq 0$. Note: a sum of about n terms, if correct, is only half of the answer. The rest of the answer is an expression for the value of the sum.

Guess: $a_n = 2^n \cdot 3 + 2^{n-1} \cdot 3 + \cdots + 2 \cdot 3 + 3$
$$= 3(1 + 2 + \cdots + 2^n)$$
$$= 3(2^{n+1} - 1)/(2-1) = 3 \cdot (2^{n+1} - 1)$$