201909 Math 122 A01 Quiz #4

#V00:__________________ Name: Solutions

This quiz has 2 pages and 5 questions. There are 15 marks available. The time limit is 25 minutes. Math and Stats standard calculators are allowed, but not needed. Except when indicated, it is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the pages for rough or extra work.

1. [2] Use the blank to indicate whether each statement is **True** (T) or **False** (F). No justification is needed. Let $A = \{1, 2, \ldots, 100\}$.

   F The number of non-empty proper subsets of A that contain the number 100 equals $2^{99} - 2$.

   F The number of subsets of A that contain 1 and do not contain 3 equals 2^{99}.

   T The number of subsets of A that contain at least 2 elements is $2^{100} - 100 - 1$.

   F If B is a set such that $|B| = 50$ and $|A \cup B| = 120$, then $|A \cap B| = 20$.

2. [5] Use induction to prove that $1 + 3 + 3^2 + \cdots + 3^n = \frac{3^{n+1}-1}{2}$ for all integers $n \geq 0$.

<u>Basis</u>: When $n=0$ the LHS $= 1$ & the RHS $= \frac{3^1-1}{2} = 1$

$\therefore$ The stmt is true when $n=0$.

<u>IH</u>: Assume there is an integer $k \geq 0$ s.t.

$$1+3+\cdots+3^n = \frac{3^{n+1}-1}{2} \quad \text{for } n = 0, 1, \ldots, k.$$

<u>IS</u>: Want $1+3+\cdots+3^{k+1} = \frac{3^{(k+1)+1}-1}{2} = \frac{3^{k+2}-1}{2}$

Look at $1+3+\cdots+3^{k+1}$

$$= \underbrace{1+3+\cdots+3^k} + 3^{k+1}$$

$$= \frac{3^{k+1}-1}{2} + \frac{2 \cdot 3^{k+1}}{2} \quad \text{by IH}$$

$$= \frac{3 \cdot 3^{k+1}-1}{2} = \frac{3^{k+2}-1}{2} \quad \text{as wanted}$$

$\therefore$ By PMI, $1+3+\cdots+3^n = \frac{3^{n+1}-1}{2} \quad \forall\, n \geq 0$

3. [1] Give a recursive definition of the sequence $s_1, s_2, \ldots$, where $s_n = 1^2 + 2^2 + \cdots + n^2$.

$$s_1 = 1$$
$$s_n = s_{n-1} + n^2$$

4. [5] Let $a_0, a_1, \ldots$ be the sequence recursively defined by $a_0 = 1$, $a_1 = 3$, and $a_n = 4a_{n-1} - 3a_{n-2}$. Use induction to prove that $a_n = 3^n$ for all integers $n \geq 0$.

<u>Basis</u> : when $n=0$, $a_0 = 1 = 3^0$
& when $n=1$, $a_1 = 3 = 3^1$.
$\therefore$ The stmt is true when $n=0$ & when $n=1$.

<u>IH</u>: Suppose there is an integer $k \geq 1$ s.t. $a_n = 3^n$ for $n = 0, 1, \ldots, k$.

<u>IS</u>: Want $a_{k+1} = 3^{k+1}$.
Look at a_{k+1}. Since $k+1 \geq 2$, we have

$$\begin{aligned} a_{k+1} &= 4a_k - 3a_{k-1} \\ &= 4 \cdot 3^k - 3 \cdot 3^{k-1} \quad \text{by IH.} \\ &= 4 \cdot 3^k - 3^k \\ &= 3 \cdot 3^k = 3^{k+1}, \text{ as wanted} \end{aligned}$$

$\therefore$ By PMI, $a_n = 3^n \; \forall n \geq 0$

5. [2] Use the blank to indicate whether each statement is **True** (T) or **False** (F). No justification is needed.

T (a) $6 + 6 \cdot 7 + 6 \cdot 7^2 + \cdots + 6 \cdot 7^{n-1} = 7^n - 1$.

F (b) For all $x, y \in \mathbb{R}$ if $x < y$ then $\lceil x \rceil \leq \lfloor y \rfloor$.

F (c) When -100 is divided by -11, the quotient equals 9 and the remainder equals -1.

F (d) The base 5 representation of 75 is $(250)_5$.

oops! my bad!