201801 Math 122 A01 Quiz #4

#V00:__________ Name: Key

This quiz has 2 pages and 5 questions. There are 15 marks available. The time limit is 25 minutes. Math and Stats standard calculators are allowed, but not really needed. Except when indicated, it is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the pages for rough or extra work.

1. [2] Let $A = \{1, 2, \ldots, 10\}$. Fill in the blanks. Do not simplify. No justification is needed.

   (a) The number of subsets of A that contain no odd integers equals 2^5.

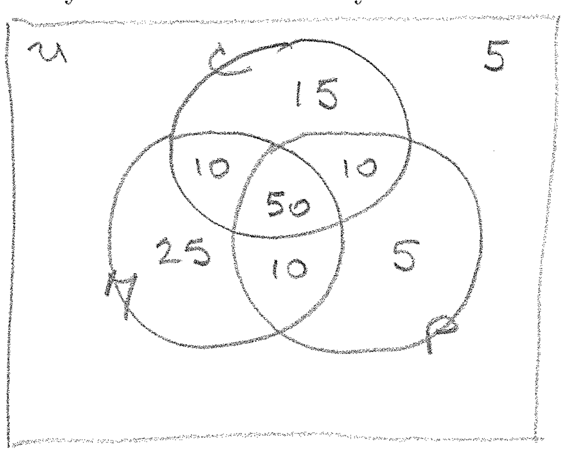
   (b) The number of nonempty, proper subsets of A equals $2^{10} - 2$.

   (c) The number of subsets of A that contain exactly one of 1, 2, 3 equals 3×2^7.

   (d) The number of subsets of A that contain 1 or 3 equals 3×2^8.

2. [2] In a group of 130 students, 85 like Computer Science, 95 like Math, and 75 like Philosophy. Any pair of these disciplines is liked by 60 students, and 50 students like all three of them.

   How many students like exactly one of the three disciplines? $15 + 25 + 5 = 45$



C = set who like CSc
M = set who like Math
P = set who like Phil.

3. [2] Fill in the blanks. No justification is necessary.

   (a) $1^2 + 2^2 + \cdots + n^2 =$ $n(n+1)(2n+1)/6$.

   (b) True or False: The Principle of Mathematical Induction implies that if the statement $S(n)$ is true for $n = k+1$ whenever it is true for the values $n = 1, 2, \ldots, k$, then the statement $S(n)$ is true for all $n \geq 1$. False.

   (c) A recursive definition of the sequence $s_0, s_1, s_2, \ldots = 1, 2, 2^2, \ldots$ is $s_0 = 1$ and, for $n \geq 1$, $s_n =$ $2s_{n-1}$.

   (d) If $a_0 = 1, a_1 = 2$ and $a_n = 3a_{n-1} + 2a_{n-2}$ for $n \geq 2$, then $a_4 =$ 100.

   $a_2 = 3 \cdot 2 + 2 \cdot 1 = 8$
   $a_3 = 3 \cdot 8 + 2 \cdot 2 = 28$
   $a_4 = 3 \cdot 28 + 2 \cdot 8 = 100$

4. [5] Use induction to prove that $2+4+\cdots+2n = n(n+1)$, for all integers $n \geq 1$. No other method will be accepted.

Basis : When $n=1$, LHS $= 2\cdot 1 = 2$ & RHS $= 1(1+1) = 2$

$\therefore$ The stmt is true when $n=1$.

IH Assume there is an integer $k \geq 1$ such that $2+4+\cdots+2n = n(n+1)$ is true when $n = 1, 2, \ldots, k$.

IS Want: $2+4+\cdots+2(k+1) = (k+1)((k+1)+1)$

Look at LHS:

$$
\begin{aligned}
&2+4+\cdots+2(k+1) \\
&= \underbrace{2+4+\cdots+2k}+2(k+1) && \text{(meaning of } \cdots \text{)} \\
&= k(k+1) + 2(k+1) && \text{by IH} \\
&= (k+1)(k+2), \text{ as wanted}
\end{aligned}
$$

$\therefore$ By induction $2+4+\cdots+2n = n(n+1)$ $\forall n \geq 1$.

5. Let $a_0, a_1, \ldots$ be the sequence defined by $a_0 = 6$ and $a_n = 4a_{n-1} + 6$ for $n \geq 1$.

(a) [2] Compute a_1, a_2, a_3, a_4. There is no need to simplify the expressions that arise.

$$
\begin{aligned}
a_1 &= 4a_0 + 6 = 4\cdot 6 + 6 \\
a_2 &= 4a_1 + 6 = 4(4\cdot 6+6)+6 \\
&= 4^2\cdot 6 + 4\cdot 6 + 6 \\
a_3 &= 4a_2 + 6 = 4(4^2\cdot 6 + 4\cdot 6 + 6) + 6 \\
&= 4^3\cdot 6 + 4^2\cdot 6 + 4\cdot 6 + 6 \\
a_4 &= 4a_3 + 6 = 4(4^3\cdot 6 + 4^2\cdot 6 + 4\cdot 6 + 6) + 6 \\
&= 4^4\cdot 6 + 4^3\cdot 6 + 4^2\cdot 6 + 4\cdot 6 + 6
\end{aligned}
$$

(b) [2] Use your work in part (a) to conjecture (ie, guess) a formula for a_n which is valid for all $n \geq 0$. Note: a sum of about n terms, if correct, is only half of the answer. The rest is an expression for the value of the sum.

Guess
$$
\begin{aligned}
a_n &= 4^n\cdot 6 + 4^{n-1}\cdot 6 + \cdots + 4\cdot 6 + 6 \\
&= 6(1+4+\cdots+4^n) \\
&= 6\,\frac{(4^{n+1}-1)}{4-1} = 3(4^{n+1}-1)
\end{aligned}
$$