202409 Quiz 5
Math 122 A04
Instructor: Jonathan Noel

Solutions

Do not open the booklet before you are told to

Date and Time: Friday, November 15, 2024 at 1:55pm.

Instructions: There are 3 pages and 5 questions. There are 15 marks available. The time limit is 25 minutes. Math and Stats standard calculators are allowed. Except when indicated, it is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the sheet for rough work.

True/False Instructions: Question 1 consists of 8 true/false questions labelled **TF 1** to **TF 8**. The last page of your test booklet is a bubble sheet for answering them. You may detach the back page from the rest of the quiz and hand it in separately if you wish. Only fill in a bubble for questions 1-8 on the bubble sheet. When making your selection, **True is A** and **False is B**. Do not select C, D or E. If you want to change your answer after filling in a bubble, then please erase your previous answer or write something on the sheet to try to make your final selection as clear as possible.

Do your rough work here. This will not be marked.

1. [4] Use the bubble sheet provided on the last page of the test booklet to indicate whether each statement is **True (A)** or **False (B)**.

   [TF 1] For all $k \in \mathbb{Z}$, if k is odd, then $2\lceil k/2 \rceil > k$. **True**

   [TF 2] The hexadecimal representation of $(111011)_2$ is $(3B)_{16}$. **True**

   [TF 3] $0 \mid 0$. **True**

   [TF 4] When -46 is divided by 5, the remainder is 1. **False**

   [TF 5] If $a > 1$ and $b > 1$ are relatively prime, then $gcd(a, b) < lcm(a, b)$. **True**

   [TF 6] If $gcd(a, b) = 4$, then there exist integers x, y such that $ax + by = 12$. **True**

   [TF 7] $9^{500} + 13^{43} + 16^{109} \equiv 0 \bmod 4$. **False**

   [TF 8] The exponent of 2 in the prime factorization of $(6!)^9$ is 36. **True**

2. [2] Find the base 7 representation of 1220.

   Suppose that
   $$1220 = d_3 \cdot 7^3 + d_2 \cdot 7^2 + d_1 \cdot 7^1 + d_0 \cdot 7^0.$$
   where $d_1, d_2, d_3 \in \{0, \ldots, 6\}$. Then
   $$1220 = 174 \cdot 7 + 2 \Rightarrow d_0 = 2$$
   $$174 = 24 \cdot 7 + 6 \Rightarrow d_1 = 6$$
   $$24 = 3 \cdot 7 + 3 \Rightarrow d_2 = 3 \text{ and } d_3 = 3.$$
   So, $(1220)_{10} = (3362)_7$.

3. [3] Let $a, b, c \in \mathbb{Z}$. Prove that if $a \mid b$ and $a \mid c$, then $a^3 \mid 4b^2c$.

   Since $a \mid b$, there exists $k \in \mathbb{Z}$ such that $ak = b$. Since $a \mid c$, there exists $\ell \in \mathbb{Z}$ such that $a\ell = c$. So,
   $$4b^2c = 4(ak)^2(a\ell) = \left(4k^2\ell\right) a^3.$$
   Since $k, \ell \in \mathbb{Z}$, we have that $4k^2\ell \in \mathbb{Z}$. Therefore $a^2 \mid 4b^2c$.

Do your rough work here. This will not be marked.

4. [2] Let $a, b \in \mathbb{Z}$ such that $20^{500}a = 14^{1324}b$. Using the Fundamental Theorem of Arithmetic, explain why b must be divisible by 5.

Since a is an integer, the prime decomposition of $20^{500}a$ contains 5 with an exponent of at least 500. By the Fundamental Theorem of Arithmetic, the prime decomposition is unique. So, since $20^{500}a = 14^{1324}b$, the prime decomposition of $14^{1324}b$ must contain 5. But $14^{1324} = 2^{1324}7^{1324}$ does not contain 5 in its prime decomposition and so b must.

5. (a) [2] Use the Euclidean algorithm to find $d = gcd(308, 854)$.

$$854 = 2 \cdot 308 + 238$$
$$308 = 1 \cdot 238 + 70$$
$$238 = 3 \cdot 70 + 28$$
$$70 = 2 \cdot 28 + 14$$
$$28 = 2 \cdot 14 + 0$$

The gcd is 14.

(b) [1] Use your work from (a) to find integers x, y such that $d = 308x + 854y$.

$$14 = 70 - 2 \cdot 28$$
$$= 70 - 2 \cdot (238 - 3 \cdot 70)$$
$$= 7 \cdot 70 - 2 \cdot 238$$
$$= 7 \cdot (308 - 1 \cdot 238) - 2 \cdot 238$$
$$= 7 \cdot 308 - 9 \cdot 238$$
$$= 25 \cdot 308 - 9 \cdot 854$$

So $x = -9$ and $y = 25$.

(c) [1] Use your answer from (a) to find $lcm(308, 854)$.

$$lcm(308, 854) = \frac{308 \cdot 854}{gcd(308, 854)} = \frac{308 \cdot 854}{14} = 18788$$

Do your rough work here. This will not be marked.