202409 Quiz 6
Math 122 A04
Instructor: Jonathan Noel

Solutions

Do not open the booklet before you are told to

Date and Time: Friday, November 29, 2024 at 1:55pm.

Instructions: There are 3 pages and 4 questions. There are 15 marks available. The time limit is 25 minutes. Math and Stats standard calculators are allowed. Except when indicated, it is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the sheet for rough work.

True/False Instructions: Question 1 consists of 8 true/false questions labelled **TF 1** to **TF 8**. The last page of your test booklet is a bubble sheet for answering them. You may detach the back page and hand it in separately if you wish. Only fill in a bubble for questions 1-8 on the bubble sheet. When making your selection, **True is A** and **False is B**. Do not select C, D or E. If you want to change your answer after filling in a bubble, then please erase your previous answer or write something on the sheet to try to make your final selection as clear as possible.

Do your rough work here. This will not be marked.

1. [4] Use the bubble sheet to indicate whether each statement is **True (A)** or **False (B)**.

   Let $\mathcal{R}$ be a relation on $A = \{1, 2, 3, 4, 5\}$.

   [TF 1] If $\mathcal{R}$ is transitive, $2\mathcal{R}3$ and $3\mathcal{R}4$, then $4\mathcal{R}2$. **False**

   [TF 2] If $\mathcal{R}$ is symmetric and antisymmetric, then $(1, 2) \notin \mathcal{R}$. **True**

   [TF 3] If $(5, 5) \notin \mathcal{R}$, then $\mathcal{R}$ is not transitive. **False**

   [TF 4] If $\mathcal{R}$ is an equivalence relation such that $(1, 2), (3, 4) \in \mathcal{R}$, then $(1, 4) \in \mathcal{R}$. **False**

   [TF 5] Every function $f : A \to A$ is a relation on A. **True**

   [TF 6] If $f : A \to A$, then $|f| = 5$. **True**

   [TF 7] If $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then $A \times B \subseteq B \times A$. **False**

   [TF 8] The set $(\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ is countable. **True**

2. Let $A = \{-100, -99, \ldots, 99, 100\}$. Let $\sim$ be the relation on A defined by $a \sim b$ if and only if $|a| = |b|$. For example, $-5 \sim 5$ because $|-5| = |5|$.

   (a) [3] Prove that $\sim$ is an equivalence relation.

   Reflexive: Every $x \in A$ satisfies $|x| = |x|$. So, $x \sim x$ for all $x \in A$.
   Symmetric: Let $x, y \in A$ such that $x \sim y$. Then $|x| = |y|$. So, $|y| = |x|$ and therefore $y \sim x$.
   Transitive: Let $x, y, z \in A$ such that $x \sim y$ and $y \sim z$. Then $|x| = |y|$ and $|y| = |z|$. Therefore, $|x| = |z|$ and so $x \sim z$.

   (b) [1] List all of the elements of [14].

   $[14] = \{-14, 14\}$.

Do your rough work here. This will not be marked.

3. [3] Provide an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ that is injective but not surjective. Prove that it has the desired properties.

   An example is $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x$. We show that it is injective. Let $x, y \in \mathbb{Z}$ such that $f(x) = f(y)$. Then $2x = 2y$ and so $x = y$. To see that it is not surjective, note that there cannot exist $x \in \mathbb{Z}$ such that $f(x) = 1$ since $f(x) = 2x$ is even and 1 is odd.

4. (a) [2] Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 7x - 2$ for every $x \in \mathbb{R}$. Prove that f is onto.

   Let $y \in \mathbb{R}$ and define $x = \frac{y+2}{7}$. Then

$$f(x) = 7x - 2 = 7\left(\frac{y+2}{7}\right) - 2 = y$$

   Therefore, f is onto.

   (b) [2] Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 2x^2 - 1$ for every $x \in \mathbb{R}$. Find $f \circ g$.

   $f \circ g(x) = f(g(x)) = 7g(x) - 2 = 7(2x^2 - 1) - 2$.

Do your rough work here. This will not be marked.