202201 Math 122 A03 Quiz #6

#V00: ________________ Name: Solutions

This quiz has 2 pages and 6 questions. There are 15 marks available. The time limit is 25 minutes. Math and Stats standard calculators are allowed, but not needed. Except when indicated, it is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the pages for rough or extra work.

1. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No reasons are necessary.

   $a=4, b=2, x=1, y=-1$; $\gcd(a,b)=2\neq 1$.

   F For non-zero integers a and b, if there are integers x, y such that $ax + by = 2$, then a and b are always relatively prime.

   $-2k+14 \equiv -2(7)+14 \equiv 0 \mod 8$

   T If $k \equiv 7 \mod 8$, then $-2k + 14$ is divisible by 8.

   T For any integer $b > 1$ and integers d_0 and d_1, $d_1b + d_0$ is divisible by $b - 1$ if and only if $d_1 + d_0$ is divisible by $b - 1$. $d_1b+d_0 \equiv d_1(1)+d_0 \mod b-1$.

   F An anti-symmetric relation $\mathcal{R}$ on $\{1, 2, 3\}$ can contain both $(1, 3)$ and $(3, 1)$.

   contradicts the definition $(1\neq 3)$.

2. [2] Prove that if $x \equiv 4 \mod 14$, then $4x + 1 \equiv 3 \mod 7$.

   Since $x\equiv 4 \mod 14$, we have $x-4=14k$ for some $k\in\mathbb{Z}$.

   So, we obtain: $4x+1 \equiv 4(14k+4)+1 \equiv 4(0+4)+1 \mod 7$

   $\equiv 16+1 \equiv 17 \equiv 3 \mod 7$,

   where we used the fact that $14k\equiv 0 \mod 7$. $\blacksquare$

3. [3] Find the ones digit of 23^{37} in base 9 without directly computing 23^{37}.

   We reduce 23^{37} mod 9!

   $(25=27-2)$

   $23^{37} \equiv (5)^{37} \equiv 5^1\cdot(5^2)^{18} \equiv 5\cdot(25)^{18} \equiv 5(-2)^{18} \mod 9$

   $(23=18+5)$

   $\equiv 5(-8)^6 \equiv 5(1)^6 \equiv 5 \mod 9$.

   Thus the last digit of 23^{37} in base 9 is $\boxed{5}$.

4. [4] Let $A = \{1,2,3,4,5\}$. The set $\mathcal{R} = \{(a,b) \in A \times A \ : \ b - a \in \{0,2\}\}$ is a relation on A. Write out $\mathcal{R}$ explicitly; then, determine how to add *as few ordered pairs as possible* to $\mathcal{R}$ to get an equivalence relation $\mathcal{E}$ on A (you don't need to write $\mathcal{E}$ out after that, just the pairs to add). How many equivalence classes does $\mathcal{E}$ have?

$\mathcal{R}$ contains the pairs (a,b) where $a,b \in A$ and $b-a=0$ or $b-a=2$. So:

$\mathcal{R} = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,3),(2,4),(3,5)\}$.

- $\mathcal{R}$ is already reflexive! ✓
- For $\mathcal{R}$ to be symmetric, we need to add $(3,1)$, $(4,2)$, and $(5,3)$.
- For $\mathcal{R}$ to be transitive, we need to further add $(1,5)$ and $(5,1)$.

Our resulting equiv. rel $\mathcal{E}$ has $\boxed{2}$ equiv. classes: $\{1,3,5\}$ and $\{2,4\}$.

5. [2] Let $B = \{a,b,c,d\}$. The set $f = \{(a,c),(b,b),(c,c),(d,a)\}$ is a function from B to B. Find counter-examples (with justification) to show that f is...

- not one-to-one:

  (a,c) and (c,c) are both in f, ie. $f(a)=f(c)$ but $a \neq c$.

  Thus f is not one-to-one.

- not onto:

  $d \in B$, but the range of f is only $\{a,b,c\}$,

  so f is not onto.

6. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No reasons are necessary.

$(1,1)$ is in both $A \times B$ and $B \times A$

_T_ For $A = \{1,2\}$ and $B = \{1,3\}$, $(A \times B) \cap (B \times A) \neq \emptyset$.

_T_ Congruence modulo 4 partitions $\mathbb{Z}$ into four equivalence classes. $\mathbb{Z} = [0] \cup [1] \cup [2] \cup [3]$

_T_ Let A be a non-empty set. Then $A \times A$ is an equivalence relation on A.

_F_ If $f : A \to B$ is one-to-one, then $A = \text{range}(f)$.

$A \times A$ contains all pairs (a,b), so all three conditions are satisfied.

range$(f) \subseteq B$, not $\subseteq A$.

eg. $f : \{1\} \to \{a\}$, $f(1) = a$.