201701 Math 122 [A01] Quiz #6

#V00:________________ Name: ______Key______

This quiz has 2 pages and 5 questions. There are 15 marks available. The time limit is 25 minutes. Math and Stats standard calculators are allowed. They might even be useful. Except when indicated, it is necessary to show clearly organized work in order to receive full or partial credit. Use the back of the pages for rough or extra work.

1. [2] Use the blank to indicate whether each statement is true or false. No reasons are necessary.

   _F_ There are integers a and b such that $gcd(a, b) = 0$.
   _T_ $gcd(2^6 5^4, 3^2 5^2) = 5^2$.
   _T_ $gcd(150, 200) \times lcm(150, 200) = 30,000$.
   _T_ If $a, b \in \mathbb{Z}$ and there are $x, y \in \mathbb{Z}$ so that $ax + by = 2$, then $gcd(a, b)$ equals 1 or 2.

2. [4] Use the Euclidean Algorithm to find $d = gcd(280, 63)$, and then use your work to find integers x and y such that $280x + 63y = d$.

   $280 = 63 \times 4 + 28$
   $63 = 28 \times 2 + 7$ ← gcd
   $28 = 4 \times 7 + 0$

   $7 = 63 - 28 \times 2$
   $= 63 - (280 - 63 \times 4) \times 2$
   $= 63 \times 9 - 280 \times 2$
   $= 280 \times \underset{x}{(-2)} + 63 \times \underset{y}{9}$

3. [2] Use the blank to indicate whether each statement is true or false. No reasons are necessary.

   _T_ $55 \times 13^4 - 94 \equiv 3 \pmod 3$
   _T_ For any positive integer n, the last digit of 31^n equals 1.
   _T_ For $b, q, r \in \mathbb{Z}$, if $n = bq + r$, then $n \equiv r \pmod b$.
   _T_ One of the integers $97, 98, \ldots, 108$ is congruent to -6 modulo 12.

4. [4] Use the Principle of Mathematical Induction to prove that, for all $n \geq 1$,

$$1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1.$$

Basis: When $n=1$, LHS $= 1 \cdot 1! = 1$
RHS $= (1+1)! - 1 = 1$

$\therefore$ The statement is true when $n=1$.

IH: For some $k \geq 1$, assume

$1 \cdot 1! = (1+1)! - 1$,

$1 \cdot 1! + 2 \cdot 2! = (2+1)! - 1$,

$\vdots$

$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1$

or just assume this.

IS: Want $1 \cdot 1! + 2 \cdot 2! + \cdots + (k+1) \cdot (k+1)! = (k+2)! - 1$

Consider $1 \cdot 1! + 2 \cdot 2! + \cdots + (k+1)(k+1)!$

$= \underbrace{1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k!}_{(k+1)! - 1} + (k+1)(k+1)!$

$= (k+1)! - 1 + (k+1)(k+1)!$

$= [1 + (k+1)](k+1)! - 1 = (k+2)! - 1$, as wanted

$\therefore$ By PMI $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1 \quad \forall n \geq 1$

5. [3] Let $a_1, a_2, \ldots$ be the sequence recursively defined by

$$a_1 = 1, \text{ and } a_n = 3a_{\lfloor n/2 \rfloor} + 1 \text{ if } n \geq 2.$$

Express each of a_1, a_2, a_4 and a_8 as a summation, and then use your work to conjecture a formula for a_{2^k}, $k \geq 0$, that does not involve a summation. A proof is not needed.

$a_1 = 1$

$a_2 = 3a_{\lfloor 2/2 \rfloor} + 1 = 3a_1 + 1 = 3 \cdot 1 + 1 \quad (= 4)$

$a_4 = 3a_{\lfloor 4/2 \rfloor} + 1 = 3a_2 + 1 =$
$= 3(3 \cdot 1 + 1) + 1 = 3^2 + 3 + 1 \quad (= 13)$

$a_8 = 3a_{\lfloor 8/2 \rfloor} + 1 = 3a_4 + 1$
$= 3(3^2 + 3 + 1) + 1 = 3^3 + 3^2 + 3 + 1 \quad (= 40)$

Guess $a_{2^k} = 3^k + 3^{k-1} + \cdots + 1$
$= (3^{k+1} - 1)/(3 - 1)$