Simultaneous Equations

\[ y_1 = \alpha_1 y_2 + \beta_1 z_1 + u_1 \]

\[ y_2 = \alpha_2 y_1 + \beta_2 z_2 + u_2 \]
Simultaneity

Simultaneity is a specific type of endogeneity problem.

Here, the explanatory variable is jointly determined with the dependent variable.

As with other types of endogeneity, OLS estimates would be biased and inconsistent.

IV estimation can be used to solve this problem.

There are some special issues to consider with simultaneous equations models (SEM).
Supply and Demand Example

Let’s start with an equation you’d like to estimate, say a labor supply function

\[ h_s = \alpha_1 w + \beta_1 z + u_1 \]

Where \( w \) is the wage and \( z \) is a supply shifter (e.g. non-labor income or number of children)

We call this a structural equation – it’s derived from economic theory and has a causal interpretation where \( w \) directly affects \( h_s \)
Example (cont)

Problem:
- We can’t just regress observed hours on wage, because observed hours and wages are determined by the equilibrium of supply and demand
- i.e. we only observe equilibrium wages
- Thus, we must also consider a second structural equation -- the labor demand function

\[ h_d = \alpha_2 w + u_2 \]
- May also have shift variables (e.g. price of capital)
- So hours are determined by a SEM
Example (cont)

- Notice that both $h$ and $w$ are endogenous because they are determined by the equilibrium of supply and demand.
- However, $z$ is exogenous.
- We need this exogenous supply shifter to allow us to identify the structural demand equation.
- With no observed demand shifters, supply is not identified and cannot be estimated.
- We can show why this is the case graphically.
Identification of Demand Equation

$D \quad S (z=z_1)$

$S (z=z_2)$

$S (z=z_3)$
Using IV to Estimate Demand

- We can, therefore, estimate the structural demand equation, using $z$ as an instrument for $w$.
- First stage equation is $w = \pi_0 + \pi_1 z + v_2$.
- Second stage equation is $h = \alpha_2 + u_2$.
- Thus, 2SLS provides a consistent estimator of $\alpha_2$, the slope of the demand curve.
- We cannot estimate $\alpha_1$, the slope of the supply curve unless we can also find a demand shifter that doesn’t belong in the supply equation.
The General SEM

More generally, suppose you want to estimate the structural equation: $y_1 = \alpha_1 y_2 + \beta_1 z_1 + u_1$

where, $y_2 = \alpha_2 y_1 + \beta_2 z_2 + u_2$

Thus, $y_2 = \alpha_2 (\alpha_1 y_2 + \beta_1 z_1 + u_1) + \beta_2 z_2 + u_2$

So, $(1 - \alpha_2 \alpha_1) y_2 = \alpha_2 \beta_1 z_1 + \beta_2 z_2 + \alpha_2 u_1 + u_2$,

We can rewrite this as the reduced form equation:

$y_2 = \pi_1 z_1 + \pi_2 z_2 + v_2$
The General SEM (continued)

Now, since \( v_2 \) is a linear function of \( u_1, y_2 \) is correlated with the error term \( (u_1) \) in the structural equation (i.e. \( y_2 \) is endogenous)

Thus, estimating the structural equation for \( y_1 \) by OLS will lead to a biased estimate of \( \alpha_1 \) – called simultaneity bias

The sign of the bias is complicated, but can use the simple regression case as a rule of thumb

In the simple regression case, the sign of the bias is the same as \( \alpha_2/(1 – \alpha_2 \alpha_1) \)
Identification of General SEM

Let $z_1$ be all the exogenous variables in the first equation, and $z_2$ be all the exogenous variables in the second equation.

It’s okay for there to be overlap in $z_1$ and $z_2$.

To identify equation 1, there must be some variables in $z_2$ that are not in $z_1$.

To identify equation 2, there must be some variables in $z_1$ that are not in $z_2$.

We refer to this as the “order condition”.
Rank and Order Conditions

Also, in order to get identification we also need to satisfy the rank condition which says more than the order condition.

The exogenous variable excluded from the first equation must also have a non-zero coefficient in the second equation for the rank condition to hold.

Note that the order condition clearly holds if the rank condition does – there will be an exogenous variable for the endogenous one.
Estimation of the General SEM

- Estimation of SEM is straightforward
- The instruments for 2SLS are the exogenous variables from *both equations*
- Can extend the idea to systems with more than 2 equations
- For a given identified equation, the instruments are all of the exogenous variables in the whole system