Introduction to Mathematical and Computer Modeling in Forestry

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Forest Management & Policy Analysis using Computer Models

1. What is the purpose?
    - Policy analysis?
    - Making forest plans (2nd guessing company plans)?
    - Informing oneself, Chief Forester or politicians?
2. How detailed are your needs?
3. How transparent should any modeling be to yourself and others?
4. How restricted is your knowledge or that of your colleagues and supervisors?
5. What resources are available?

Uses of computer models in forestry and ecosystem management

1. Prescriptive: suggest practical solutions for solving problems
2. Predictive: predict consequences of government policies (e.g., CRAM, integrated assessment models)
3. Sensitivity analysis: Explore alternative and extremes – 'what if' scenarios

Calibration and Verification

- Scientific forecasting procedures, e.g., as laid out by the International Institute of Forecasters (www.forecastingprinciples.com)
- Calibration
  - Positive mathematical programming (Howitt)
  - ‘Mixes’ method (McCarl and colleagues)
    - Latest advance enables inclusion of new options, land uses, management strategies, etc.
- Verification
  - Compare model outcomes to realization (ex post analysis)
  - Is back-casting possible?

Two Basic Model Types

1. Optimization models
   – Constrained mathematical programming models
      • LP, QP, NLP, DP, SDP
      • IP, MIP, MINLP
      • Fuzzy LP, MODM (goal programming), etc.
   – Advantages:
      • Solution is optimal
      • One can get **shadow prices**
      • Results (and steps) have an economic interpretation
   – With new computing power, huge problems can be addressed (>10 million constraints)
   – Can incorporate risk and risk preferences

Two Basic Models (cont)

2. Heuristics
   - Major benefit: They work, they provide an answer
   - No guarantee solution is better than ANY alternative
     - Intuition may be preferred: the on-site expert may do better than the modeler
   - Heuristics can help forest-level, on-the-ground managers design forest management plans
   - No ability to calibrate such models
   - Economists generally eschew (oppose?) such models
   - Are they useful for designing and analyzing forest policy??

Heuristics vs Optimization Models

- Pukkala & Heinonen (*Nonlinear Analysis: Real World Applications* 2006): need heuristic approach for forest planning involving multiple objectives and parties, non-linear, non-additive and spatial components
- Boston & Bettinger (*For Sci* 1999; *Silva Fenn* 2001) show that heuristics needed when dealing with spatial problems (e.g., green-up and adjacency) – John Nelson's work
- Vanderkam et al. (*Biological Conservation* 2007) show LP preferred to heuristic algorithms for designing efficient conservation reserve networks
- Williamson et al. (Ch 15 in *Environmental Modeling for Sustainable Regional Development*, 2011) demonstrate that LP is the primary method and tool for risk analysis in forestry

Question to Ask: Heuristics vs Optimization

- When do you use which approach? SOME ANSWERS
  - Rely on optimization approaches whenever possible as these are richer in various ways:
    - Easier and able to calibrate and verify
    - Results (including intermediary results) have a much richer interpretation
  - Rely on heuristics when the problem is simply too complex to solve using an optimization approach.
    - Road construction and green-up & adjacency are classic examples (spatial!)
- **Rule of Thumb**: Rely on optimization, even linear approximations of nonlinear problems, unless you are forced to use a heuristic. Even then there are heuristics that seek optimal solutions, most notably, learning models, TABU search and even fuzzy optimization methods.

General Mathematical Programming Formulation

Optimize **F(x)**

Subject to (s.t.) **G(x) $\in$ S$_1$ and $x \in$ S$_2$**

F(x), G(x) linear & x non-negative → linear program (LP)

F(x) and/or G(x) nonlinear & x non-negative → nonlinear program (NLP)

F(x) quadratic and G(x) linear & x non-negative → quadratic program (QP)

F(x) and G(x) linear and/or nonlinear & x integer → integer program (IP)

Linear Programming: Motivating Example

Poet with woodlot needs extra earnings, but wants to work no more than 180 days per year. Can earn $90/ha/yr 'managing' cedar, $120/ha/yr managing hardwoods (mixed, northern). Need 2 work days (wd) per ha per yr to 'manage' cedar; 3 wd/ha/yr for hardwoods. Poet's problem looks like this:

$$\max \quad Z = 90x_1 + 120x_2 \qquad \text{revenue}$$

$$\$/\text{y} = (\$\,\text{ha}^{-1}\text{y}^{-1})(\text{ha}) + (\$\,\text{ha}^{-1}\text{y}^{-1})(\text{ha})$$

Subject to:

$$2x_1 + 3x_2 \leq 180 \qquad \text{time constraint}$$

$$(\text{wd ha}^{-1}\text{y}^{-1})(\text{ha}) + (\text{wd ha}^{-1}\text{y}^{-1})(\text{ha}) = \text{wd/y}$$

$$x_1 \leq 40 \qquad \text{ha of cedar}$$

$$x_2 \leq 50 \qquad \text{ha of hardwood}$$

$$x_1, x_2 \geq 0 \qquad \text{non-negativity}$$

Linear Program (LP):

$$\max \quad Z = 90x_1 + 120x_2 \qquad \text{(revenue)}$$

$$s.t. \quad 2x_1 + 3x_2 \leq 180 \qquad \text{(time constraint)}$$

$$x_1 \leq 40 \qquad \text{(cedar area constraint)}$$

$$x_2 \leq 50 \qquad \text{(hardwood area constraint)}$$

$$x_1, x_2 \geq 0 \qquad \text{(non-negativity)}$$

GENERAL FORMULATION:

Max Z = c X

s.t. AX $\leq$ b

X $\geqslant$ 0

where c, b and X are vectors and A is the technical coefficients matrix

LP example: Pulp mill pollution problem

Let x_1= mechanical pulp (t/day) and x_2= chemical pulp (t/day)

Both require 1 work day per 1 t of pulp produced

BOD= Biochemical Oxygen Demand (a measure of pollution)

1 t mechanical pulp produces 1 unit BOD

1 t chemical pulp produces 1.5 units BOD.

Revenues: mechanical pulp: $100/t, chemical pulp: $200/t

Possible Objectives: *minimize BOD output*

maximize employment

maximize revenue

Constraints:

at least 300 workers need to be employed

minimum revenue of $40,000 per day

$$\min \quad Z = x_1 + 1.5x_2 \qquad \text{pollution}$$

$$\text{BOD/d} = (\text{BOD/t})(\text{t/d}) + (\text{BOD/t})(\text{t/d})$$

$s.t.$

$$x_1 + x_2 \geq 300 \qquad \text{employment constraint}$$

$$(\text{wd/t})(\text{t/d}) + (\text{wd/t})(\text{t/d}) = \text{wd/d}$$

$$100x_1 + 200x_2 \geq 40000 \quad \text{revenue constraint}$$

$$(\$/\text{t})(\text{t/d}) + (\$/\text{t})(\text{t/d}) = \$/\text{d}$$

$$\left.\begin{aligned} x_1 &\leq 300 \\ x_2 &\leq 200 \end{aligned}\right\} \qquad \text{capacity constraints}$$

$$x_1, x_2 \geq 0 \qquad \text{non} - \text{negativity}$$

Multiple-objective decision making (MODM) problem

$$\min Z = x_1 + 1.5x_2$$

$$
\begin{aligned}
s.t.\ \ & x_1 + x_2 \geq 300 \\
& 100x_1 + 200x_2 \geq 40000 \\
& x_1 \leq 300 \\
& x_2 \leq 200 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Multiply both sides by -1 to get into standard form:

$$\max\,(-Z) = -x_1 - 1.5x_2$$

$$
\begin{aligned}
s.t.\ \ & -x_1 - x_2 \leq -300 \\
& -100x_1 - 200x_2 \leq -40000 \\
& x_1 \leq 300 \\
& x_2 \leq 200 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Standard form of LP problem:

Max $\quad Z = c\,X \quad$ (n decision variables)

s.t. $\quad AX \le b \quad$ (m constraints)

$\quad X \ge 0$

where $c_{1\times n} = [c_1, c_2, ..., c_n]$

$$X_{n\times 1} = \begin{bmatrix} X_1 \\ X_2 \\ \cdots \\ \cdots \\ X_n \end{bmatrix},\; b_{m\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_m \end{bmatrix},\; A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Assumptions of LP:

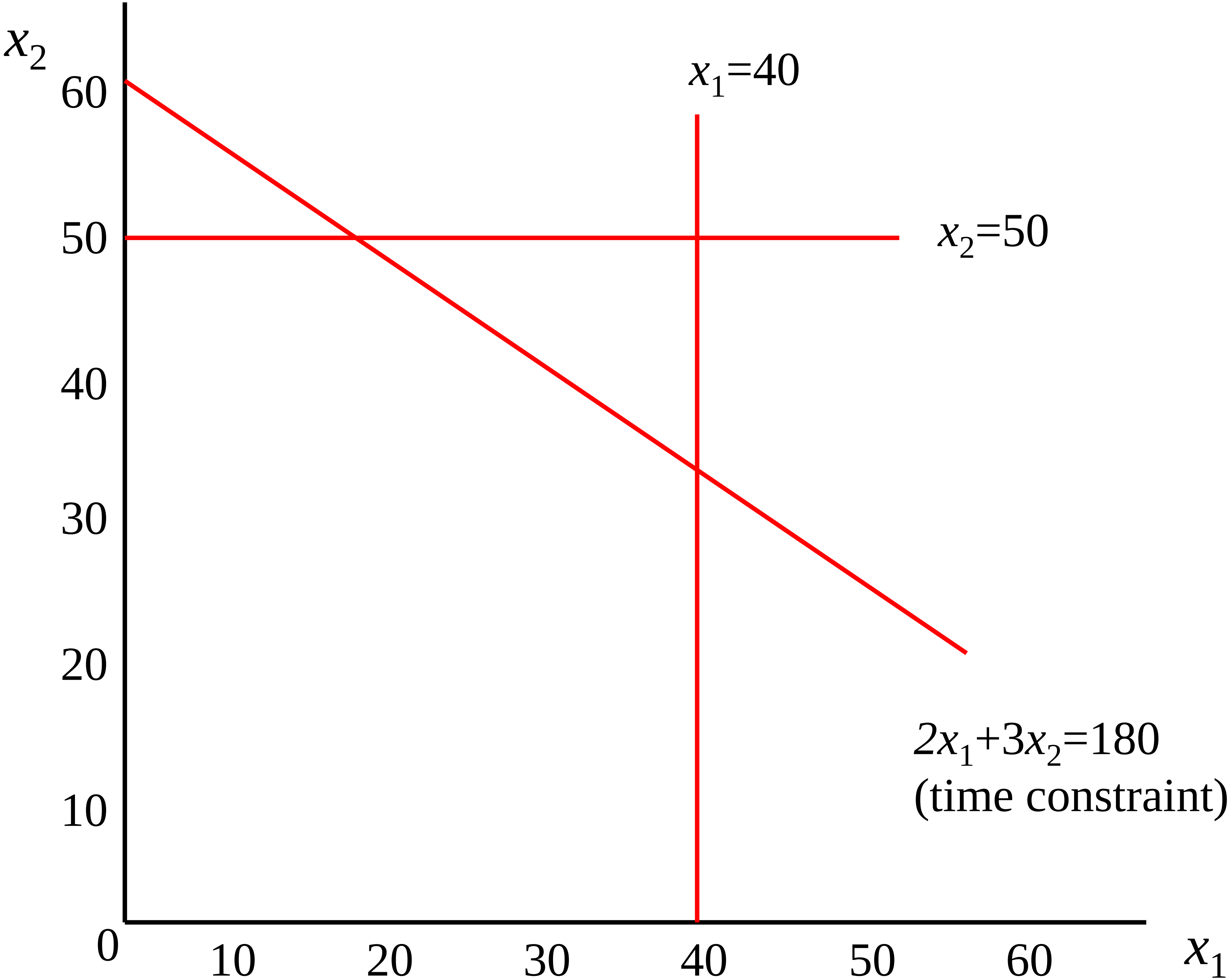
1. Objectives and constraints are appropriate to the problem at hand
2. Proportionality
   - Contribution of each decision variable to objective is constant and independent of variable level
   - Use of each resource per unit of each decision variable is constant and independent of variable level

   NO ECONOMIES OF SCALE
3. Addititivity (not multiplicative, no interactions)
4. Divisibility (decision variables infinitely divisible)
5. Certainty: There is no stochasticity/randomness

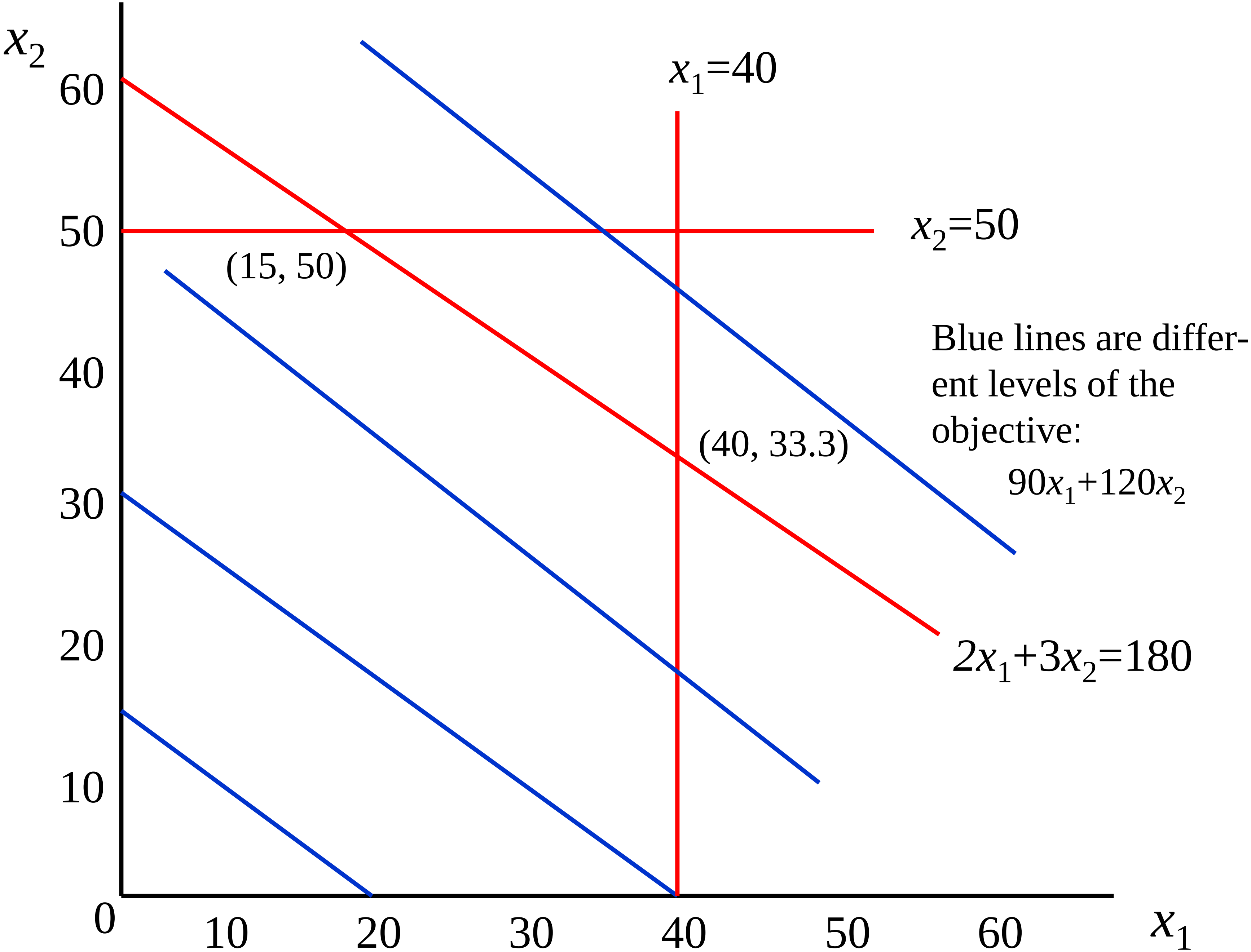
Solving LPs: Graphical Solution

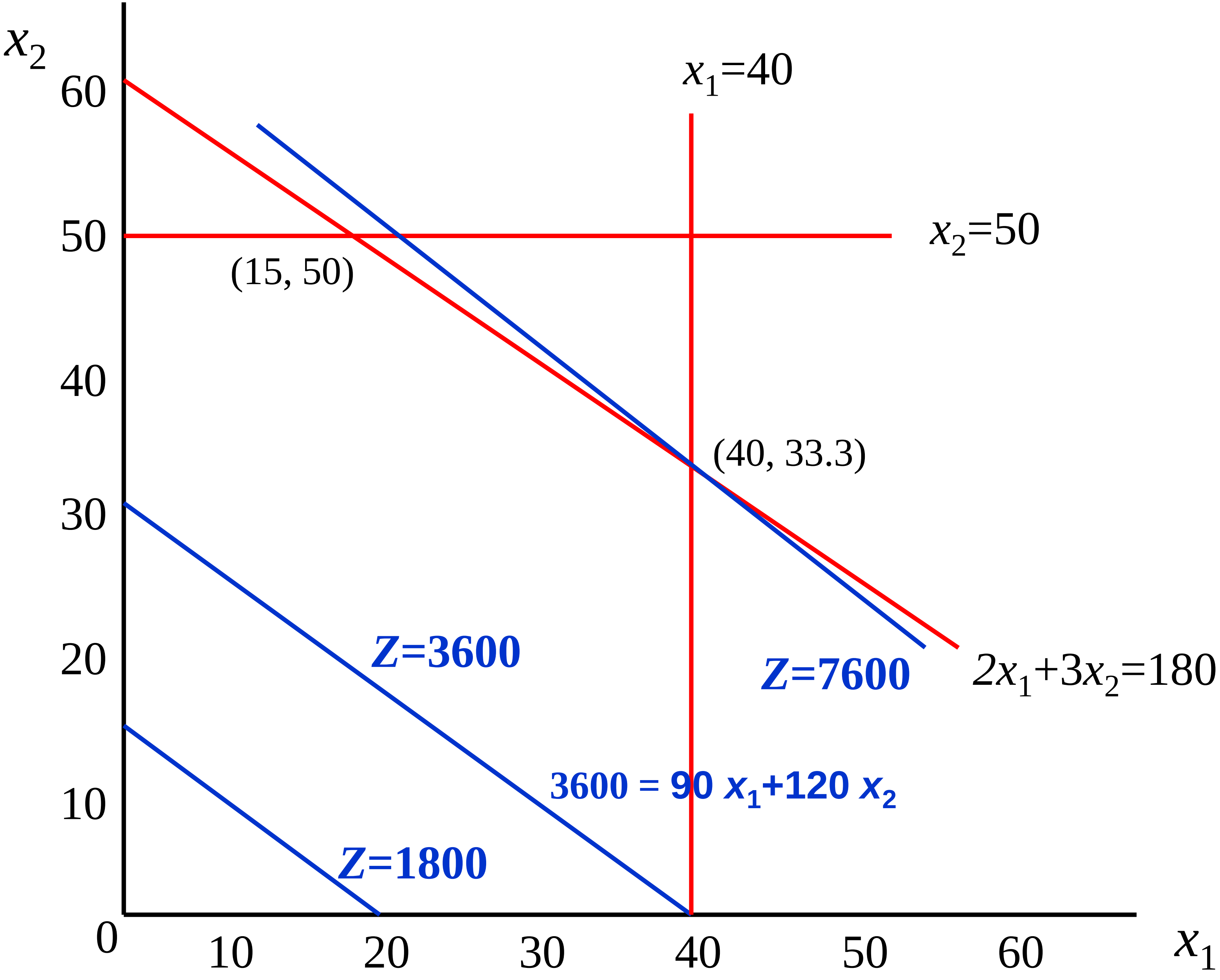
Consider Again the Poet Problem:

Max	$Z = 90\, x_1 + 120\, x_2$	(revenue)
s.t.	$2\, x_1 + 3\, x_2 \leq 180$	(time)
	$x_1 \leq 40$	(ha of cedar)
	$x_2 \leq 50$	(ha hardwood
	$x_1, x_2 \geq 0$	(non-negativity)



x_2
60
50
40
30
20
10
0
10
20
30
40
50
60
x_1
$x_1=40$
$x_2=50$
$2x_1+3x_2=180$
(time constraint)


x_2
60
50
40
30
20
10
0
10
20
30
40
50
60
x_1
x_1=40
x_2=50
(15, 50)
(40, 33.3)
Blue lines are different levels of the objective:
$90x_1+120x_2$
$2x_1+3x_2=180$



x_2
60
50
40
30
20
10
0
10
20
30
40
50
60
x_1
x_1=40
x_2=50
(15, 50)
(40, 33.3)
Z=7600
$2x_1+3x_2=180$
Z=3600
3600 = 90 x_1+120 x_2
Z=1800

Further Example

A firm produces wheat and canola using tractors, land and labor as inputs.

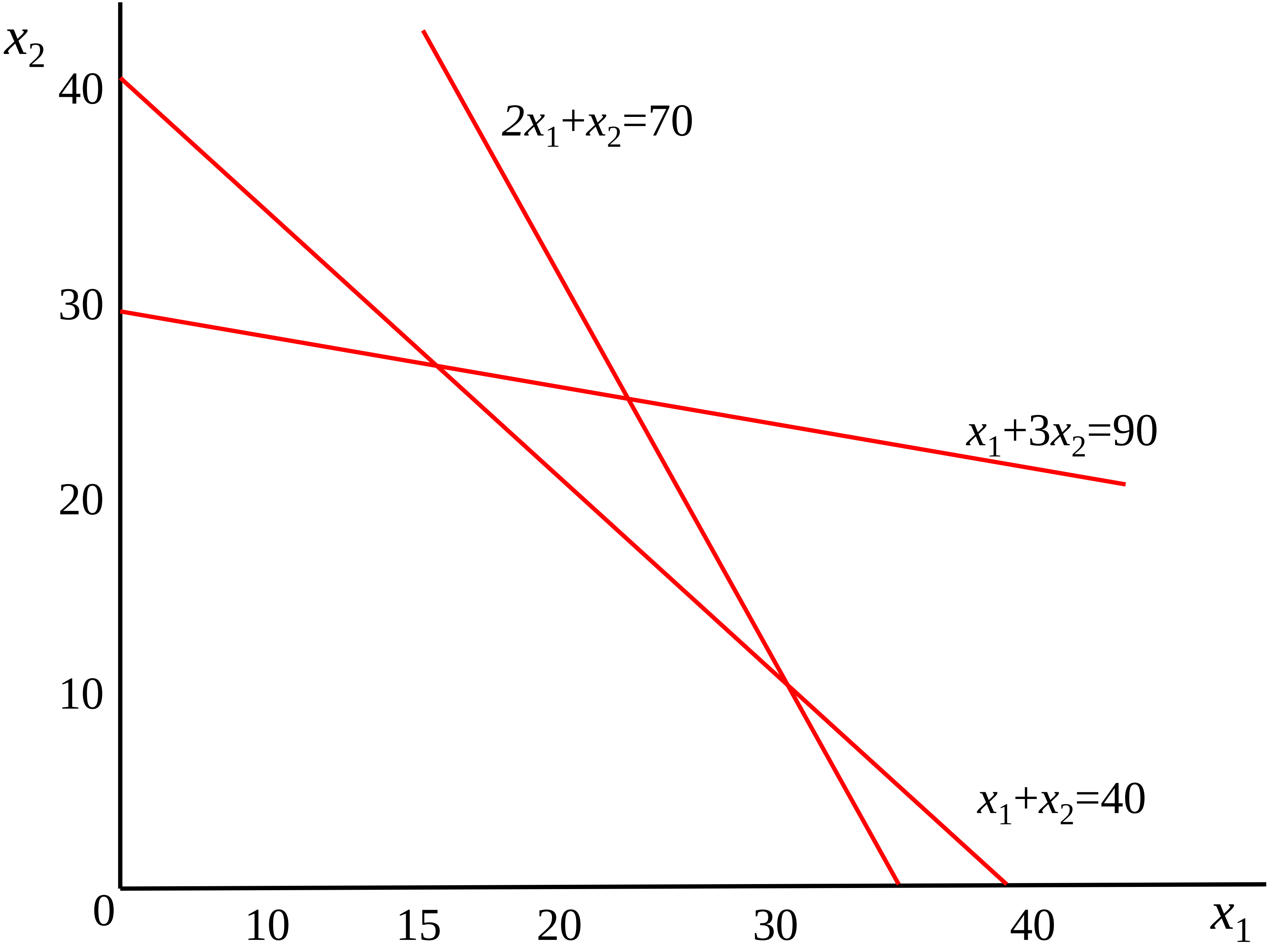
Constraint	Input Requirements		Input Availability
	Wheat	Canola	
Tractor	2	1	70
Land	1	1	40
Labor	1	3	90
Net revenue	$40	$60	

Max $\pi = 40\,x_1 + 60\,x_2$ (net revenue)

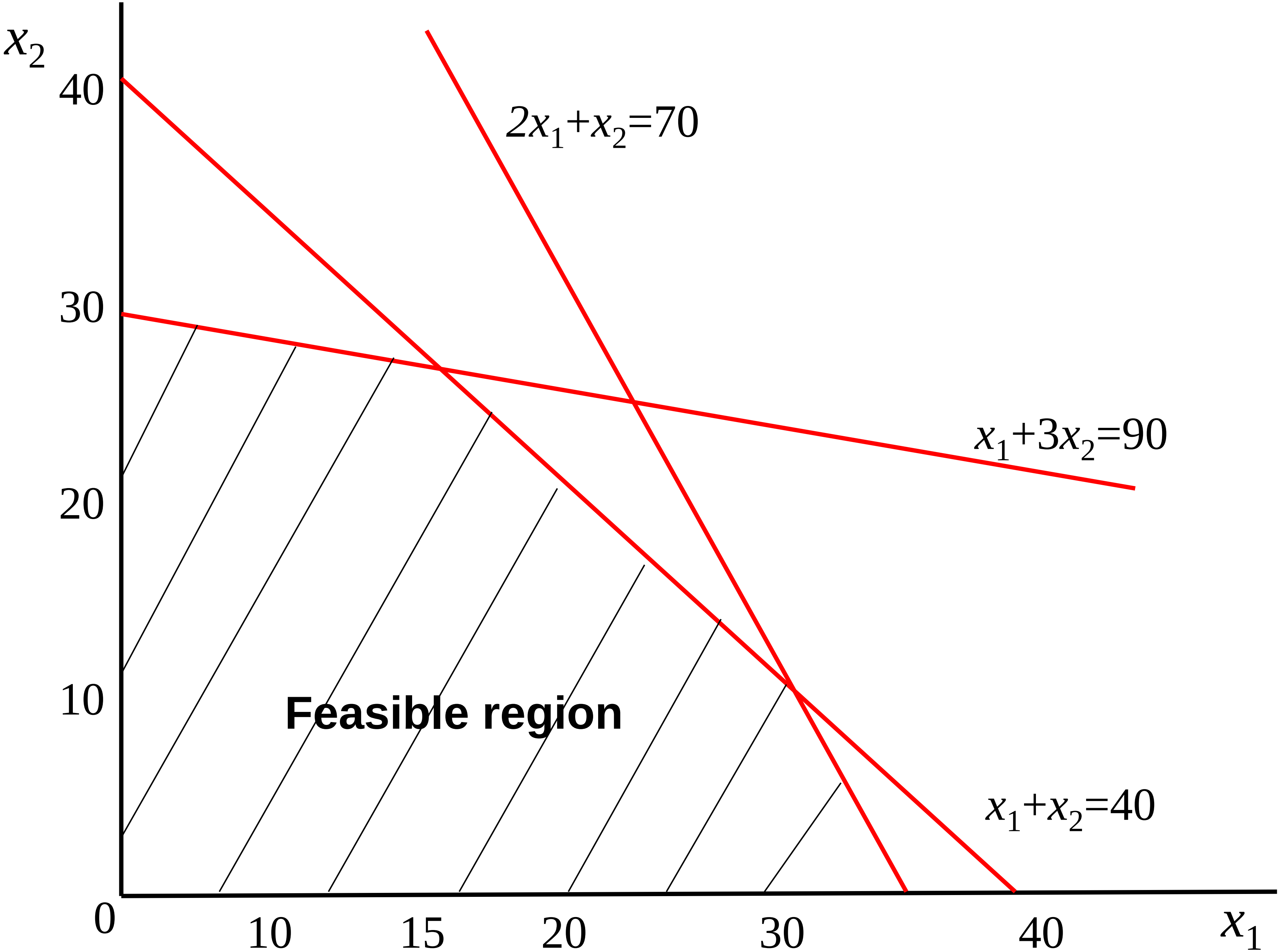
s.t.

$2\,x_1 + x_2$	≤ 70	(tractor hours)
$x_1 + x_2$	≤ 40	(land in ha)
$x_1 + 3\,x_2$	≤ 90	(labor hours)
$x_1, x_2 \geq 0$		(non-negativity)

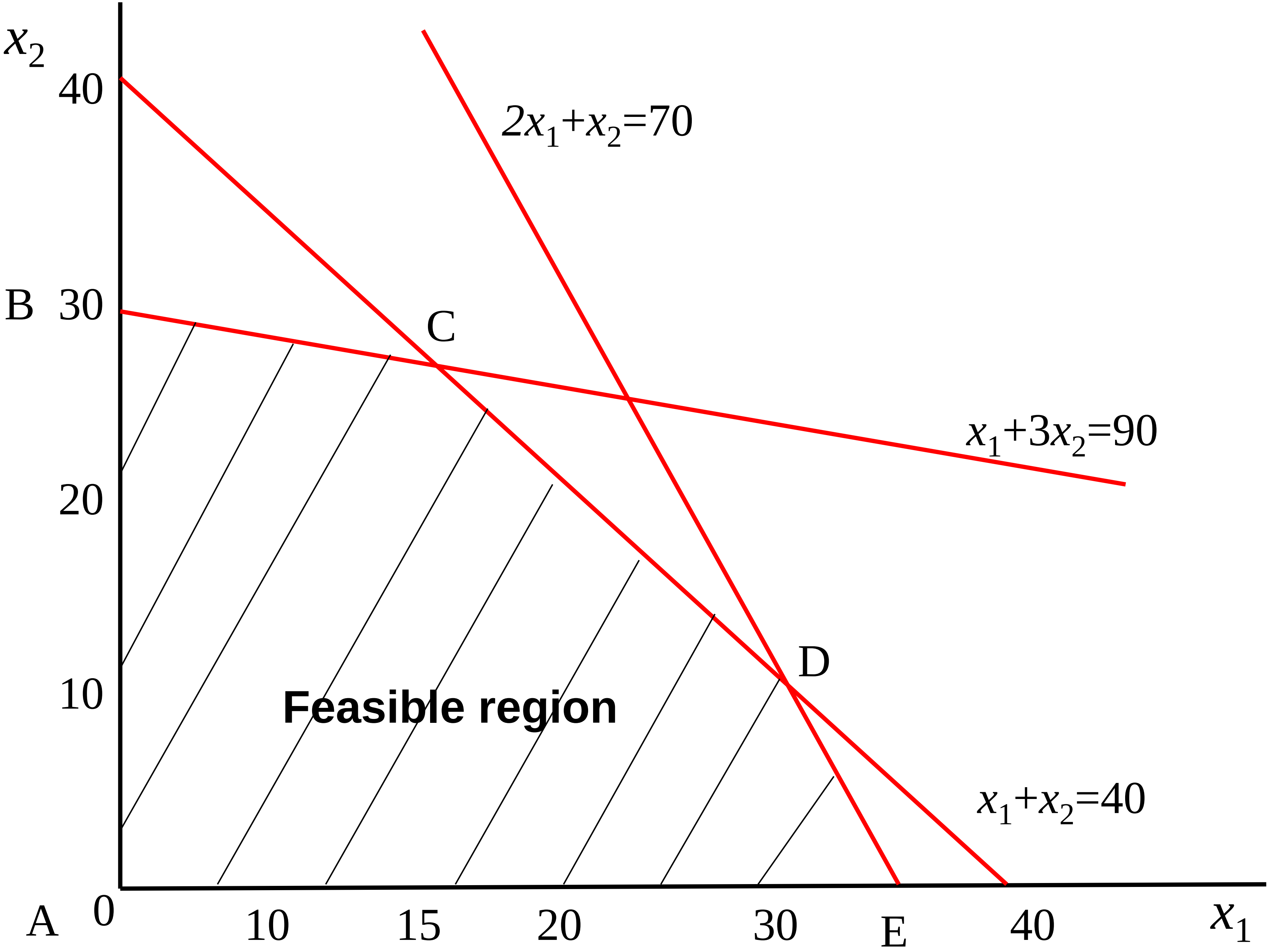
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix},\ b = \begin{bmatrix} 70 \\ 40 \\ 90 \end{bmatrix},\ c = [40\ 60],\ X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



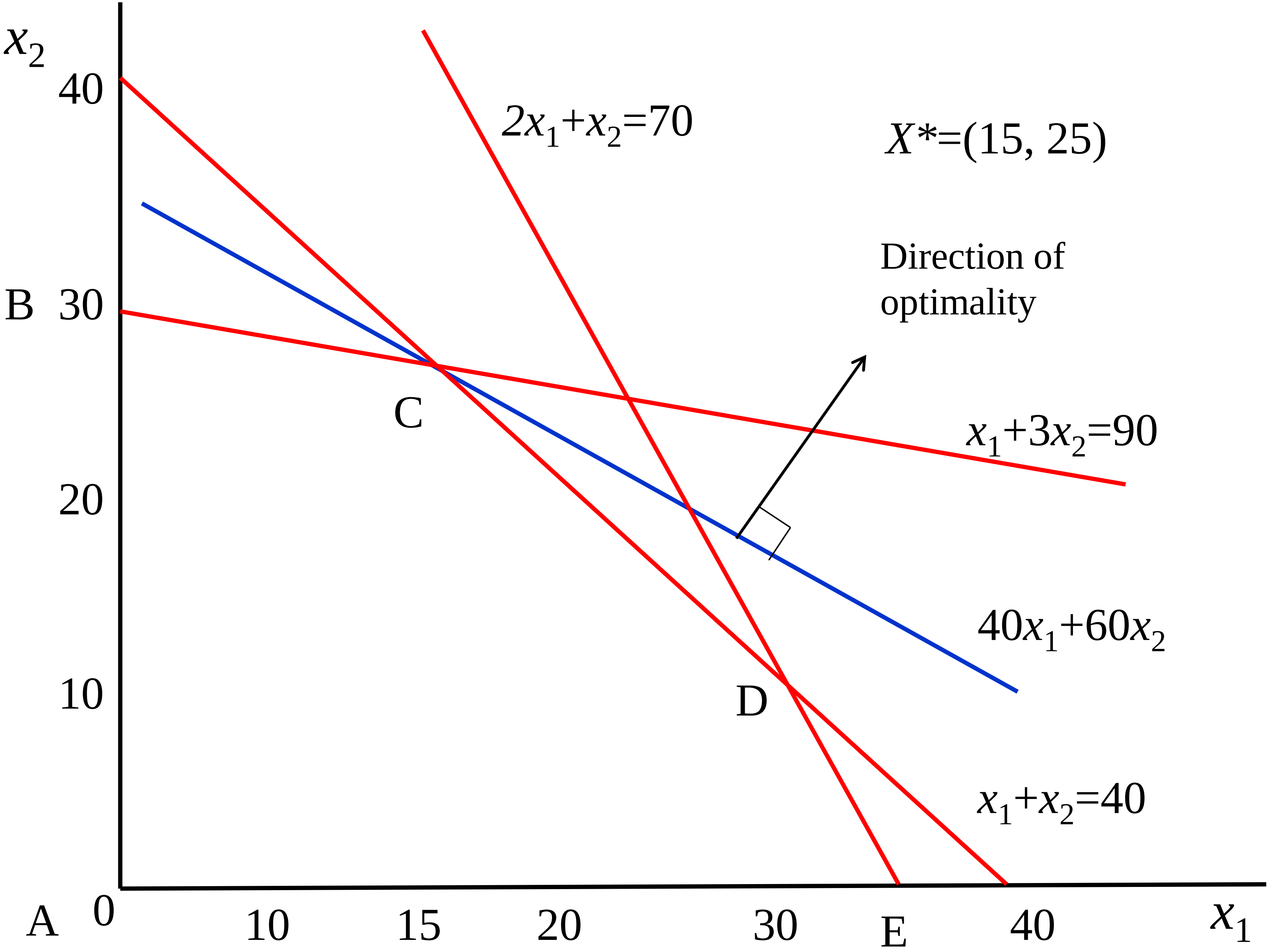
x2
40
30
20
10
0
10
15
20
30
40
x1
2x1+x2=70
x1+3x2=90
x1+x2=40



x_2
40
30
20
10
0
$2x_1+x_2=70$
$x_1+3x_2=90$
$x_1+x_2=40$
Feasible region
10
15
20
30
40
x_1



x_2
40
$2x_1+x_2=70$
B
30
C
$x_1+3x_2=90$
20
D
10
Feasible region
$x_1+x_2=40$
A
0
10
15
20
30
E
40
x_1



x_2
40
30
B
20
10
A
0
10
15
20
30
E
40
x_1
$2x_1+x_2=70$
$X^*=(15, 25)$
Direction of optimality
C
$x_1+3x_2=90$
$40x_1+60x_2$
D
$x_1+x_2=40$

Computational Software

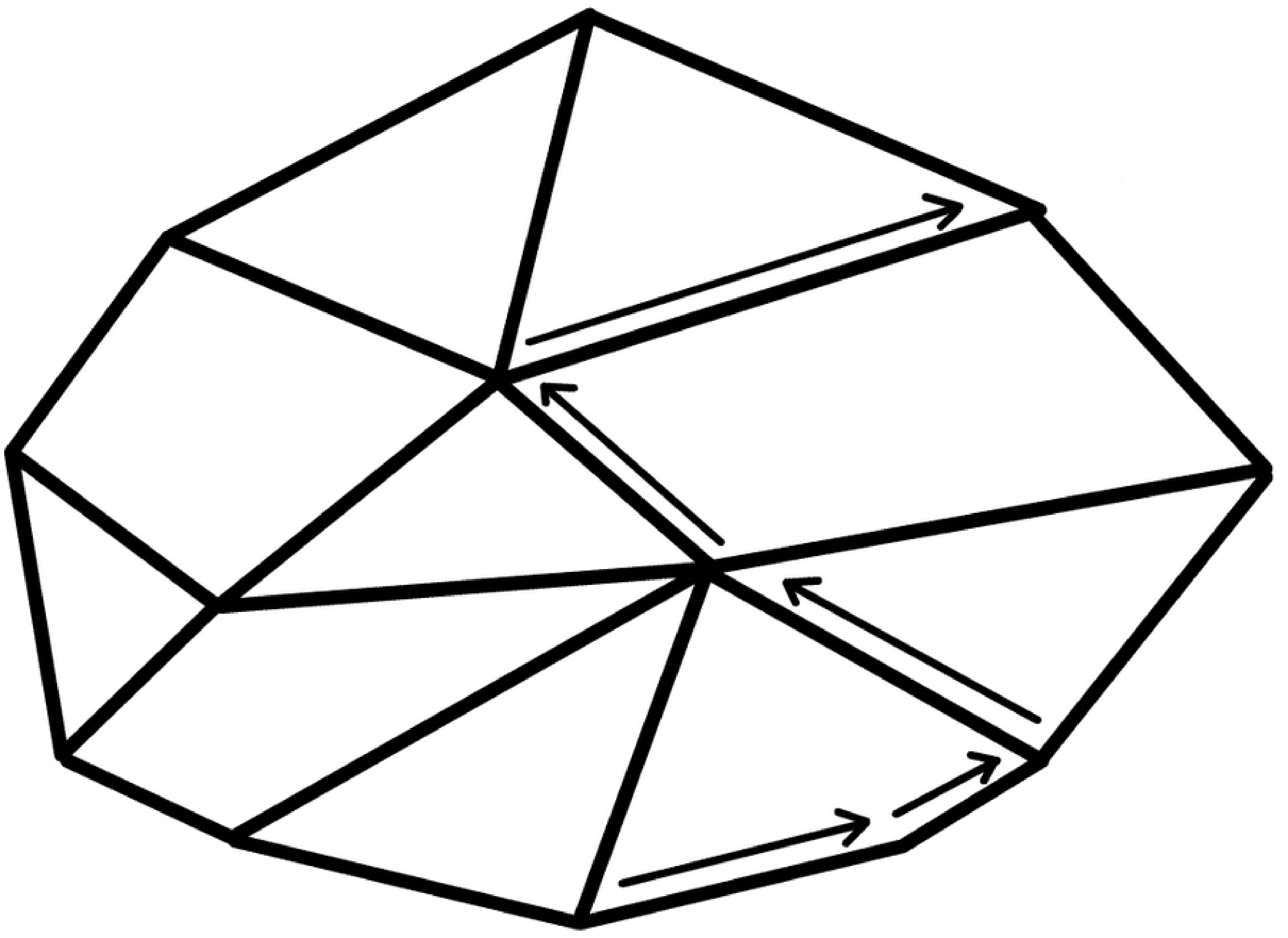
- Excel (Solver) (Can imbed LP in a program such as VBA)
- GAMS
- Matlab (can call GAMS from within Matlab)
- Other
  - XA (Add-in to Excel or stand alone)
  - Premium Solver (add-in to Excel)

Points:

- LP solution algorithms are pretty standard and based on **simplex algorithm**
- Lots of different solvers for non-linear (NLP), integer (IP), mixed-integer (MIP), etc. However, QP and many NLP problems are solved using the simplex algorithm simply by taking nonlinear constraints and making them into linear pieces. Think of a soccer ball – it is not truly round but consists of many planes

Simplex Algorithm

('The algorithm that controls your life')



Slack variable representation of PRIMAL agricultural problem:

Max $\pi = 40\, x_1 + 60\, x_2 + 0x_{s1} + 0x_{s2} + 0\mathrm{x}_{s3}$ (net revenue)

s.t.

$$
\begin{aligned}
2\, x_1 + x_2 + x_{s1} &= 70 && \text{(tractor hours)} \\
x_1 + x_2 + x_{s2} &= 40 && \text{(land in ha)} \\
x_1 + 3\, x_2 + x_{s3} &= 90 && \text{(labor hours)} \\
x_1, x_2, x_{s1}, x_{s2}, x_{s3} &\geq 0 && \text{(non-negativity)}
\end{aligned}
$$

Begin with slack variables set to RHS constraint values

Duality

- For every PRIMAL problem, there is a DUAL problem
- Solving the PRIMAL simultaneously solves the DUAL (i.e., slack variables), and vice versa
- If a solution to the PRIMAL cannot be achieved, it may be possible to get a solution by solving the DUAL instead
- For economic applications, the DUAL variables have an important interpretation as shadow prices

Duality (cont)

PRIMAL		**DUAL**	
Max	$Rev = c\,X$	Min	$Cost = b\,Y$
s.t.	$AX \leq b$	s.t.	$A'\,Y \geq \mathrm{c}$
	$X \geq 0$		$Y \geq 0$

Maximize	$\leftrightarrow$	Minimize
$\leq$ constraint	$\leftrightarrow$	$y \geq 0$
$x \geq 0$	$\leftrightarrow$	$\geq$ constraint
$=$ constraint	$\leftrightarrow$	y free
x free	$\leftrightarrow$	$=$ constraint

Duality (cont)

PRIMAL		DUAL
Max NR = $30x_1 + 45x_2$	$\leftrightarrow$	Min TC = $15y_1 + 10y_2 + 0y_3$
Subject to	$\leftrightarrow$	Subject to
input use $\leq$ input supply	$\leftrightarrow$	imputed input price ≥ 0
$4x_1 + 3x_2 \leq 15$	$\leftrightarrow$	$y_1 \geq 0$
$2x_1 + 1x_2 \leq 10$	$\leftrightarrow$	$y_2 \geq 0$
$-x_1 + 5x_2 \leq 0$	$\leftrightarrow$	$y_3 \geq 0$
$x_1 \geq 0$	$\leftrightarrow$	$4y_1 + 2y_2 - y_3 \geq 30$
$x_2 \geq 0$	$\leftrightarrow$	$3y_1 + y_2 + 5y_3 \geq 45$
Activity levels ≥ 0	$\leftrightarrow$	MC $\geq$ MR

DUAL slack representation of agricultural problem:

Min $C = 70\,y_1 + 40\,y_2 + 90y_3 + 0y_{s1} + 0y_{s2}$ (cost)

s.t.

$$2\,y_1 + y_2 + y_3 - y_{s1} = 40 \quad \text{(wheat)}$$

$$y_1 + y_2 + 3y_3 - y_{s2} = 60 \quad \text{(canola)}$$

$$y_1, y_2, y_3, y_{s1}, y_{s2} \geq 0 \quad \text{(non-negativity)}$$

y_{s1} and y_{s2} are the dual slack (or surplus) variables: y_{s1} is marginal loss for wheat; y_{s2} is marginal loss for canola

Slack & Dual Slack (Surplus) Variables

From two slides earlier, we had MC ≥ MR in the dual representation. The constrains of the dual problem can be associated with a slack (or marginal loss) variable, which has the following definition and meaning:

MC = MR + dual slack variable

MC – marginal loss = MR

where the marginal loss is identically equal to the dual slack variable. That is why we subtract y_s from the left-hand-side of the dual constraints in the previous slide.

If you are dealing with an economics problem (where the objective is to minimize cost or maximize net private or social benefits), then every ‘move’ has an economic interpretation

	Primal Variables	Primal Slack Variables	
Z	$x_1 \cdots\cdots x_n$	$x_{s1} \cdots\cdots x_{sm}$	P_{sol}
0	$MRTT_{11} \cdots MRTT_{1n}$	$MRTT_{1s1} \cdots MRTT_{1sm}$	x_{B1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	$MRTT_{m1} \cdots MRTT_{mn}$	$MRTT_{ms1} \cdots MRTT_{msn}$	x_{Bm}
1	$(Z_1-C_1) \cdots (Z_n-C_n)$	$(Z_{s1}-C_{s1}) \cdots (Z_{sm}-C_{sm})$	Z
D_{sol}	$y_{s1} \cdots\cdots y_{sn}$	$y_1 \cdots\cdots y_m$	R
	Dual Slack Variables	Dual Variables	

APPENDIX A
SIMPLEX ALGORITHM

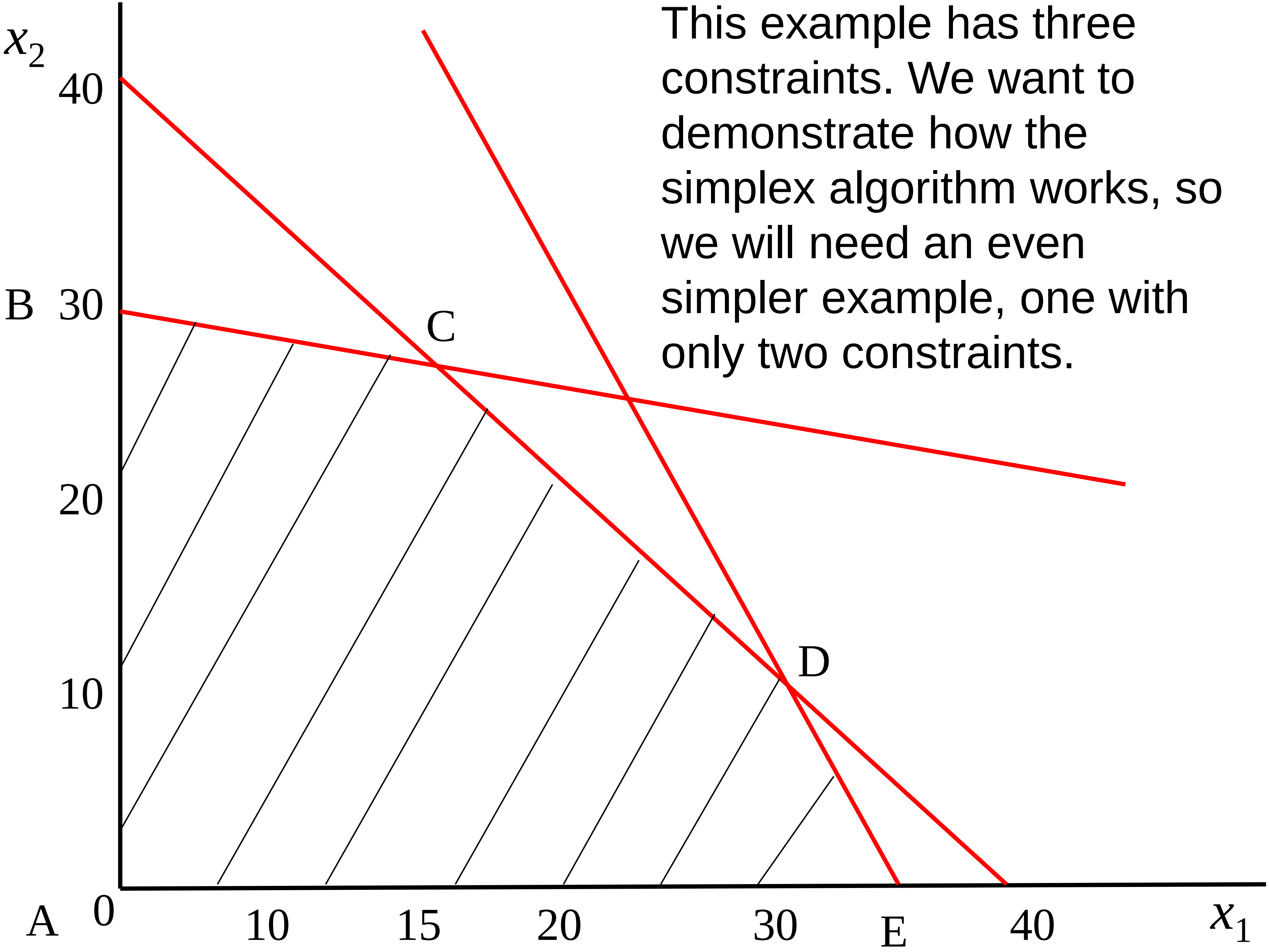
- How does it work?
- What does it mean?

Primal Simplex Algorithm

Step 1: Know solution lies at extreme point, choose a feasible basis and compute corresponding basic feasible solution. We begin with point A = (0,0) – the origin where all the slack variables equal RHS values of the constraint.

Step 2: Verify if the basic feasible solution from Step 1 is also an optimal solution. If 'yes', stop; if not, continue.

Step 3: Select an adjacent extreme point (new basic feasible solution) by changing old feasible basis by only one column vector (go from A to B or A to E – see next slide). Go to Step 2.



This example has three constraints. We want to demonstrate how the simplex algorithm works, so we will need an even simpler example, one with only two constraints.

Wheat (x_1) & Corn (x_2) Output Optimization:

Max	$Z = 3x_1 + 5x_2$	(revenue)
s.t.	$2x_1 + 4x_2 \leq 16$	(land constraint)
	$6x_1 + 3x_2 \leq 18$	(labor constraint)
	$x_1, x_2 \geq 0$	(non-negativity)

Introduce Slack Variables:

Max	$Z = 3x_1 + 5x_2 + 0x_{s1} + 0x_{s2}$	(revenue)
s.t.	$2x_1 + 4x_2 + x_{s1} = 16$	(land)
	$6x_1 + 3x_2 + x_{s2} = 18$	(labor)
	$x_1, x_2, x_{s1}, x_{s2} \geq 0$	(non-negativity)

Some asides:

- In the above, wheat and corn are measured in the same units, say tons. Thus, the amount of land needed to grow a unit of corn is greater than that needed per unit of wheat
- Note: In the final solution, a slack variable may be > 0. Thus, since not all of the resource is used, its shadow value is 0. A general rule: $p_s \times x_s = 0$ (**Complementary slackness condition**)
  - Either there is no unused resource or its shadow value is zero, or both

Step 1: Starting feasible basic solution is:

$$x_1 = 0,\ x_2 = 0,\ x_{s1} > 0,\ x_{s2} > 0$$

Step 2: Calculate value of Z (which = 0 at this point)

Entry Criterion: Select activity associated with the most positive coefficient in the objective function.

$$\rightarrow x_2 \text{ enters the basis}$$

Step 3: Shift the basic feasible solution from one extreme point (feasible basis) to an adjacent one.

Activity x_2 enters the basis, so one of x_{s1} or x_{s2} must exit. Determining which to exit is a decision with economic meaning.

SUMMARY

Basic Feasible Solution

Current	New
$x_1 = 0$	$x_1 = 0$
$x_2 = 0$	$x_2 > 0$ (enters)
$x_{s1} = 16$	$x_{s1} = 16 - 4x_2 \geq 0$
$x_{s2} = 18$	$x_{s2} = 18 - 3x_2 \geq 0$

Minimum Ratio Criterion:

$x_2 = \min\{16/4, 18/3\} = \min\{4, 6\} = 4$

Therefore, new BFS solution is:

$x_1 = 0, x_2 = 4, x_{s1} = 0, x_{s2} = 6$

Exit Criterion: To change the current feasible basis, eliminate the column corresponding to the index in the numerator of the minimum ratio:

$$x_{new} = \min\left\{\frac{\text{value of current feasible solution}}{\text{positive coefficients of new activity}}\right\}$$

Take x_{s1} out of the basis and replace it with x_2.

Iteration #2

Step 2: Evaluate current basic feasible solution.

$Z = 3\,x_1 + 5\,x_2 + 0x_{s1} + 0x_{s2} = 3(0) + 5(4) + 0(0) + 0(6) = 20$

Corn is in the basis. All land is used up, but there is slack labor. Remaining candidate is wheat but, to get wheat in, some corn acreage must be given up. So must determine the opportunity cost (OC) of wheat in terms of corn. **Recall:**

$$y = f(x_1, x_2) \rightarrow \mathrm{d}y = f_1\,\mathrm{d}x_1 + f_2\,\mathrm{d}x_2$$

Along a production possibility frontier:

$$\mathrm{d}y = 0 \quad \rightarrow f_1\,\mathrm{d}x_1 + f_2\,\mathrm{d}x_2 = 0$$

$$\rightarrow \mathrm{MRTT}_{x2,\,x1} = -\mathrm{d}x_2/\mathrm{d}x_1 = f_1/f_2$$

$$= \text{Marginal sacrifice}_1/\text{Marginal sacrifice}_2$$

Constraint on land: $2x_1 + 4x_2 + x_{s1} = 16$

Total differentiating:

$2\,dx_1 + 4\,dx_2 = 0$ (as x_{s1} is taken as constant)

$\rightarrow \text{MRTT}_{x2 \rightarrow x1} = -\,dx_2/dx_1 = ½$

A generalized matrix version of this is:

Max cX s.t. $A\,x_1 + B\,x_2$

$\rightarrow \text{MRTT}_{x2 \rightarrow x1} = -\,dx_2/dx_1 = B^{-1}A$

Recall our problem: Wheat (x_1) & Corn (x_2) Output Optimization

Max	$Z = 3\,x_1 + 5\,x_2$	(revenue)
s.t.	$2\,x_1 + 4\,x_2 \leq 16$	(land constraint)
	$6\,x_1 + 3\,x_2 \leq 18$	(labor constraint)
	$x_1, x_2 \geq 0$	(non-negativity)

<u>Opportunity marginal cost:</u> Sacrifice of one additional unit of output as measured by the foregone alternative production opportunity, which is measured by the MRTT (slope of the transformation frontier).

A sacrifice can be positive or negative. Avoid positive sacrifice; welcome negative sacrifice.

OC of wheat (x_1)

= (sacrifice in terms of corn) – (unit revenue of wheat)

= (wheat land/corn land) × (revenue of 1 unit corn land)

– (revenue of one unit wheat land)

$$= (2/4) \times 5 - 3 = -\tfrac{1}{2}$$

Note in the previous slide that:

$MRTT_{\text{wheat land}\rightarrow\text{corn land}}$ = wheat land/corn land
= 2/4

(Note that MRTT is constant, which is why it is simply a ratio, because of LP assumptions)

To improve returns, therefore, redistribute available resources from corn to wheat production since the sacrifice is negative (i.e., there is a benefit from so doing) as $OC_{wheat}<0$.

<u>Step 3:</u> We showed wheat enters. What activity should leave?

<u>Opportunity input requirement</u> of a given commodity (wheat) is the savings (as opposed to sacrifice) of inputs attributable to one unit of a foregone activity (corn) adjusted by the MRTT.

Recall: Land constraint is fully satisfied.

Opportunity labor requirement of wheat

= (wheat labor requirement) – (saving in terms of corn)

$= 6 - \frac{1}{2} \times 3 = 9/2$

Labor requirement

$\text{MRTT}_{x2 \to x1 \text{ in land}}$ since land is limiting

Labor saving

Current BFS	**New BFS**
$x_1 = 0$	$x_1 > 0$
$x_2 = 4$	$x_2 = 4 - ½x_1 \geq 0$
$x_{s1} = 0$	$x_{s1} = 0$
$x_{s2} = 6$	$x_{s2} = 6 - (9/2)\ x_1 \geq 0$

Calculations:

$4 \geq ½\ x_1 \rightarrow x_1 = 8$; $6 \geq 9/2\ x_1 \rightarrow x_1 = 4/3$

$x_1 = \min\{8, 4/3\} = 4/3$

Then: $x_1 = 4/3$, $x_2 = 10/3$, $x_{s1} = 0$, $x_{s2} = 0$

Iteration #3

Step 2: $Z = 3(4/3) + 5(10/3) + 0(0) + 0(0) = 62/3$

Corn and wheat are in the BFS and there is no unused land or labor. Thus, this appears to be the optimal solution.

Entry Criterion: Select activity associated with most negative OC. (There is none!)

Exit Criterion: Eliminate column corresponding to index in numerator of the minimum ratio. (Not needed)

Recall our problem: Wheat (x_1) & Corn (x_2) Output Optimization

Max	$Z = 3x_1 + 5x_2$	(revenue)
s.t.	$2x_1 + 4x_2 \leq 16$	(land constraint)
	$6x_1 + 3x_2 \leq 18$	(labor constraint)
	$x_1, x_2 \geq 0$	(non-negativity)

Setting it up in tableau format:

Initial Primal Tableau:

Z	x_1	x_2	x_{s1}	x_{s2}	Sol
0	2	4	1	0	16
0	6	3	0	1	18
1	–3	–5	0	0	0

Step 2 (exit):
Minimum ratio criterion

x_{s1}: 16/4 = 4 → Exit x_{s1}

x_{s2}: 18/3 = 6

Step 1 (entry): Choose most negative value in bottom row for entry so x_2 enters

Pivot element is circled and must be positive as must all elements in shaded column above the bottom row.

First Iteration:
Use row operations to get new tableau

Initial tableau that needs to be changed

Z	x_1	x_2	x_{s1}	x_{s2}	Sol
0	2	4	1	0	16
0	6	3	0	1	18
1	–3	–5	0	0	0

Multiply first row by ¼ to get 1 in the column under x_2

Z	x_1	x_2	x_{s1}	x_{s2}	Sol
0	1/2	1	1/4	0	4

Use row operations to get new tableau

Now use row operations so that remaining entries in x_2 column are 0.
New 2nd row = old 2nd row – 3 $\times$ new 1st row
New 3rd row = old 3rd row + 5 $\times$ new 1st row.

Z	x_1	x_2	x_{s1}	x_{s2}	Sol
0	½	1	1/4	0	4
0	9/2	0	-3/4	1	6
1	-½	0	5/4	0	20

Step 2 (exit): Minimum ratio criterion

x_2: 4/½ = 8

x_{s2}: 6 / 9/2 = 4/3 → Exit x_{s2}

Step 1 (entry): Choose most negative value in bottom row for entry so x_1 enters

Second Iteration:
Use row operations to get new tableau

Second tableau that now needs to be changed

Z	x_1	x_2	x_{s1}	x_{s2}	Sol
0	½	1	1/4	0	4
0	9/2	0	-3/4	1	6
1	-½	0	5/4	0	20

Multiply second row by 2/9 to get 1 in the column under x_1

Z	x_1	x_2	x_{s1}	x_{s2}	Sol
0	1	0	-1/6	2/9	4/3

Second Iteration:
Use row operations to get new tableau

Now use row operations so that remaining entries in x_2 column are 0.

New 1st row = old 1st row – ½ × new 2nd row

New 3rd row = old 3rd row + ½ × new 2nd row.

Z	x_1	x_2	x_{s1}	x_{s2}	Sol
0	½	1	1/4	0	4
0	9/2	0	-3/4	1	6
1	-½	0	5/4	0	20

Old tableau after 1st iteration

Z	x_1	x_2	x_{s1}	x_{s2}	Sol
0	0	1	1/3	1/9	10/3
0	1	0	-1/6	2/9	4/3
1	0	0	7/6	1/9	20 2/3

New tableau after 2nd iteration

Final Solution

Note: All of the entries in the final (objective function) row are positive, so the entry requirement says no new variable will enter the basic feasible solution. Therefore, this is the solution.

Z	x_1	x_2	x_{s1}	x_{s2}	Sol
0	0	1	1/3	1/9	10/3
0	1	0	-1/6	2/9	4/3
1	0	0	**7/6**	**1/9**	20 2/3
	y_{s1}	y_{s2}	$\mathbf{y_1}$	$\mathbf{y_2}$	

Dual and dual slack variables are indicated, with y_1 and y_2 the shadow prices (dual variables).

Possibilities:

- Optimal solution is found.
- Unbounded solution. Value of primal objective increases without bound. Occurs when it is not possible to find pivot in entering column because all elements ≤ 0.
- Infeasibility of one or more constraints. Constraints are inconsistent and there is no feasible solution to the problem – this is a frequent result.
- Degeneracy occurs if there are redundant constraints (e.g., $x_1 \leq 25$, $x_2 \leq 25$ and $x_1 + x_2 \leq 50$)
- More than one optimal (a variable is brought into basis without increasing the objective value)

Dual Interpretation

$$MRTT_{x_2,x_{s1}} = \frac{1}{3},\ MRTT_{x_2,x_{s2}} = -\frac{1}{9},\ MRTT_{x_1,x_{s1}} = -\frac{1}{6},\ MRTT_{x_1,x_{s2}} = \frac{2}{9}$$

OC of land = (sacrifice in terms of corn and wheat) − (unit revenue of land)

= (unit revenue of foregone activities) × (marginal rates of technical transformation)

$$= \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{6} \end{bmatrix} = \frac{5}{3} - \frac{3}{6} = \frac{7}{6}$$

land(x_{s1}) ↵

$$\text{OC of labor} = \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \end{bmatrix} = -\frac{5}{9} + \frac{6}{9} = \frac{1}{9}$$

labor(x_{s2}) ↵

Opportunity costs are <u>imputed marginal values</u>, <u>dual variables</u> or <u>shadow prices</u> = value of marginal products

= MR $\times$ MP

Original Problems:

$$\text{PRIMAL: } \max\ \pi = 3x_1 + 5x_2$$

$$s.t.\ \ 2x_1 + 4x_2 \le 16$$

$$6x_1 + 3x_2 \le 18$$

$$x_1, x_2 \ge 0$$

$$\text{DUAL: } \min C = 16y_1 + 18y_2$$

$$s.t.\ \ 2y_1 + 6y_2 \ge 3$$

$$4y_1 + 3y_2 \ge 5$$

$$y_1, y_2 \ge 0$$

$$OC_{j^{th}\text{ activity}} = (\text{unit revenue of foregone activities})(\text{MRTTs})$$

$$- (\text{unit revenue of } j^{th} \text{ activity})$$

$$= Z_j - C_j$$

	Primal Variables	Primal Slack Variables	
Z	$x_1 \cdots\cdots x_n$	$x_{s1} \cdots\cdots x_{sm}$	P_{sol}
0	$MRTT_{11} \cdots MRTT_{1n}$	$MRTT_{1s1} \cdots MRTT_{1sm}$	x_{B1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	$MRTT_{m1} \cdots MRTT_{mn}$	$MRTT_{ms1} \cdots MRTT_{msn}$	x_{Bm}
1	$(Z_1-C_1) \cdots (Z_n-C_n)$	$(Z_{s1}-C_{s1}) \cdots (Z_{sm}-C_{sm})$	Z
D_{sol}	$y_{s1} \cdots\cdots y_{sn}$	$y_1 \cdots\cdots y_m$	R
	Dual Slack Variables	Dual Variables	

Dual Simplex Algorithm

Return to the Initial Primal Tableau:

$$\begin{array}{cccccc} Z & x_1 & x_2 & x_{s1} & x_{s2} & Sol \end{array}$$

$$\begin{bmatrix} 0 & 2 & 4 & 1 & 0 & 16 \\ 0 & 6 & 3 & 0 & 1 & 18 \\ 1 & -3 & -5 & 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & a_{ij}^{*} & x_{Bi} \\ 1 & Z_j - C_j & Z \end{bmatrix}$$

$$\text{Dual:}$$

$$\min R = 16y_1 + 18y_2$$

$$s.t. \;\; 2y_1 + 6y_2 \geq 3$$

$$4y_1 + 3y_2 \geq 5$$

$$y_1, y_2 \geq 0$$

$$\text{Mult by } (-1):$$

$$\max\ -R = -16y_1 - 18y_2$$

$$s.t.\ -2y_1 - 6y_2 \le -3$$

$$-4y_1 - 3y_2 \le -5$$

$$y_1, y_2 \ge 0$$

$$\begin{matrix} -R & y_1 & y_2 & y_{s1} & y_{s2} & Sol \end{matrix}$$

$$\begin{bmatrix} 0 & -2 & -6 & 1 & 0 & -3 \\ 0 & -4 & -3 & 0 & 1 & -5 \\ 1 & 16 & 18 & 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & -a_{ji}^{*} & Z_j - C_j \\ 1 & x_{Bi} & -R \end{bmatrix}$$

There are cases where it is not possible to solve the primal problem but, by going to the dual, it is possible to solve the problem using the **dual simplex method** – illustrated in the next slide.

Primal Algorithm:

$$\begin{bmatrix} 0 & a_{ij}^{*} & x_{Bi} \\ 1 & \underline{Z_j - C_j} & Z \end{bmatrix} \rightarrow 2^{nd} \text{ step: choose } \min\left\{ \frac{x_{Bi}}{a_{ij}^{*}} \middle| a_{ij}^{*} > 0 \right\}$$

$\uparrow$

(Exit Criterion)

1^{st} step: choose most negative $(Z_j - C_j)$.(Entry Criterion)

Dual Algorithm:

$$\begin{bmatrix} 0 & -a_{ji}^{*} & Z_j - C_j \\ 1 & \underline{x_{Bi}} & -R \end{bmatrix} \rightarrow 1^{st} \text{ step: choose most negative } x_{Bi}.\text{(Entry Criterion)}$$

$\downarrow$

2^{nd} step: choose $\min\left\{ \frac{Z_j - C_j}{-a_{ji}^{*}} \middle| -a_{ji}^{*} > 0 \right\}$ (Exit Criterion)

How does this work when the values in the bottom row are all non-negative? Consider:

$$
\begin{array}{cccccc}
-R & y_1 & y_2 & y_{s1} & y_{s2} & Sol
\end{array}
$$

$$
\begin{bmatrix}
0 & -2 & -6 & 1 & 0 & -3 \\
0 & -4 & -3 & 0 & 1 & -5 \\
1 & 16 & 18 & 0 & 0 & 0
\end{bmatrix}
$$

Because all $(z_j - c_j) \geq 0$, we have a dual feasible solution. So we begin, under *Sol*, by choosing the most negative for exit, and use the minimum ratio criterion for determining what y to enter.

$$
\min\left(\frac{18}{-(-3)}, \frac{16}{-(-4)}\right) = 4
$$

Compare the dual simplex algorithm on the previous slide with the original tableau and approach used by the simplex algorithm (below):

Z	x_1	x_2	x_{s1}	x_{s2}	Sol
0	2	4	1	0	16
0	6	3	0	1	18
1	–3	–5	0	0	0

Step 2 (exit): Minimum ratio criterion

x_{s1}: 16/4 = 4 → Exit x_{s1}

x_{s2}: 18/3 = 6

Step 1 (entry): Choose most negative value in bottom row for entry so x_2 enters

Here the pivot element must be positive, but in the dual simplex algorithm it must be negative as shown in the previous slide.

DUAL – PRIMAL Commonality

Primal Lagrangian:

$$L^P = 3x_1 + 5x_2 + y_1(16 - 2x_1 - 4x_2) + y_2(18 - 6x_1 - 3x_2)$$

Dual Lagrangian:

$$L^D = 16y_1 + 18y_2 + x_1(3 - 2y_1 - 6y_2) + x_2(5 - 4y_1 - 3y_2)$$

$$= 3x_1 + 5\,x_2 + y_1(16 - 2x_1 - 4x_2) + y_2(18 - 6x_1 - 3x_2)$$

Dual and Primal are bound together by a common Lagrangian.

DUAL/PRIMAL Solutions

1) If solution to primal is unique, non-degenerate and optimal, optimal solution to dual is unique
2) When primal has degenerate solution, dual has multiple optimal solutions
3) When primal has multiple optimal solutions, optimal dual solution is degenerate
4) When primal problem unbounded, dual is infeasible
5) When primal is infeasible, dual is unbounded or infeasible

APPENDIX B
Linear Programming Extensions

- Kuhn-Tucker conditions
- Sensitivity analysis
- Artificial variables method
- Big M method
- Phase I – Phase II method

Kuhn–Tucker Conditions and LP

Max $z = c'x$

s.t. $Ax \le b$

$x \ge 0$

$$L = c'x + y'(b - Ax)$$

$$\frac{\partial L}{\partial x} = c - y'A = 0$$

$$\frac{\partial L}{\partial y} = b - Ax = 0$$

Min $R = yb$

s.t. $A'y' \ge c$

$y \ge 0$

$$L = yb + x'(c - A'y')$$

$$\frac{\partial L}{\partial y} = b - Ax = 0$$

$$\frac{\partial L}{\partial x} = c' - y'A = 0$$

$c_{n \times 1}$ $x_{n \times 1}$ $b_{m \times 1}$

$A_{m \times n}$ $A'_{n \times m}$

$y_{1 \times m}$ $y'_{m \times 1}$ $c'_{1 \times m}$

Kuhn–Tucker Conditions:

(1) $c' - y'A \leq 0$

(2) $(c' - y'A)\,x = 0$

(3) $x \geq 0$

(4) $b - Ax \geq 0$ (these are just the constraints)

(5) $(b - Ax)\,y = 0$

(6) $y \geq 0$

K-T conditions are necessary and sufficient conditions for an optimal.

(2) implies that $(c' - y'A) = 0$ or $x = 0$ or both

(5) implies that $(b - Ax) = 0$ or $y = 0$ or both.

These are referred to as **Complementary Slackness** conditions.

Sensitivity Analysis

- Idea is to examine the range over which the optimal solution still applies. Range is determined by changes in:
  - Coefficients in objective function (*c* vector)
  - Values of constraints (RHS or *b* vector)
  - Changes in the technical coefficients (*A* matrix)
- GAMS, Matlab, Maple and Excel provide some of this information, but it is best to change parameter values and re-run the optimization model.

Some extensions to the Simplex method

If the LP problem is in standard form with only $\leq$ constraints and *b* values that are all positive, then there is no problem with the simplex method achieving a solution.

Problem: If there are $\leq$, $\geq$ and $=$ constraints and/or some *b* values are negative. Then we need to use the **Big M** or **Phase I/Phase II** method that involves use of **Artificial variables**.

Artificial Variables

- Slack variables are added if ≤ constraint
- Surplus variables are subtracted if ≥ constraint
- Artificial variables are added to each constraint not satisfied if the xs equal zero:
  - If the RHS of a ≤ constraint is negative, an artificial variable is added in addition to the slack variable on the LHS
  - If an = constraint, an artificial variable is added.
  - To a ≥ constraint an artificial variable is added (since a surplus variable was subtracted)
- Why? Needed to ensure a non-negative initial feasible basis

Big M method

- All artificial variables need to be driven out of the solution before we get a true feasible basis.
- The Big M method does this by adding an arbitrarily large negative penalty on the artificial variables in the objective function (recall that coefficients on slack and surplus variables are zero)
- If the artificial variables cannot be driven from the solution if the penalty is set sufficiently large, then the problem is infeasible.

Phase I/Phase II method

- Used by computer algorithms. Solvers automatically put in the slack, surplus and artificial variables; in other cases, an interior point algorithm or mixed interior-simplex method is employed.
- Phase I: Replace objective function with sum of artificial variables and minimize this sum. If the minimized objective is non-zero, the solution is infeasible
- Phase II: Use the basis found in phase I as the start of the simplex algorithm.