Quadratic Programming

G. Cornelis van Kooten

REPA Group

University of Victoria

Problem specification in matrix notation:

Max $F(x) = c\,x + x'\,\Omega\,x$

s.t. $A\,x \leq b$

$x \geq 0$

where $x'\Omega x$ is the quadratic form.

For a maximum, the objective function must be concave; for a minimum it must be convex

Concave $\rightarrow$ Ω is negative definite or negative semi-definite

Convex $\rightarrow$ Ω is positive definite or positive semi-definite

Example

$$\begin{aligned} \max \quad & 4x + 6y - 2x^2 - 2xy - 2y^2 \\ \text{s.t.} \quad & x + 2y \le 2 \\ & x, y \ge 0 \end{aligned}$$

$$c = \begin{bmatrix} 4 & 6 \end{bmatrix},\ x = \begin{bmatrix} x \\ y \end{bmatrix},\ A = \begin{bmatrix} 1 & 2 \end{bmatrix},\ b = \begin{bmatrix} 2 \end{bmatrix}$$

$$\Omega = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix},\ |-2| < 0,\ \begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} = 3 > 0 \Rightarrow \text{negative definite}$$

$$\mathrm{L} = 4x + 6y - 2x^2 - 2xy - 2y^2 + \lambda(2 - x - 2y) + \mu_1 x + \mu_2 y$$

$$(1)\ \frac{\partial L}{\partial x} = 4 - 4x - 2y - \lambda + \mu_1 = 0$$

$$(2)\ \frac{\partial L}{\partial y} = 6 - 2x - 4y - 2\lambda + \mu_2 = 0$$

$$(3)\ \frac{\partial L}{\partial \lambda} = 2 - x - 2y \ge 0 \Rightarrow x + 2y - 2 \le 0$$

$(4)\ \lambda(x+2y-2)=0$

$(5)\ \mu_1 x=0, \mu_2 y=0$

$(6)\ x, y, \lambda, \mu_1, \mu_2 \geq 0$

We convert this problem to one that can be solved using the simplex method:

Convert results (1)–(3) as follows:

$(1')\ 4x+2y+\lambda-\mu_1+A_1=4$

$(2')\ 2x+4y+2\lambda-\mu_2+A_2=6$

$(3')\ x+2y+S=2$

where A_1 and A_2 are artificial variables that are the first to be driven out of the basic feasible solution.

S is a slack variable, as usual; (4) and (5) are complementary slackness conditions.

Restate the problem as:

$$\min A_1 + A_2 = \max(-A_1 - A_2)$$

This implies:

$$\begin{aligned}
\max \quad & Z = -10 + 6x + 6y + 3\lambda - \mu_1 - \mu_2 \\
\text{s.t.} \quad & 4x + 2y + \lambda - \mu_1 + A_1 = 4 \\
& 2x + 4y + 2\lambda - \mu_2 + A_2 = 6 \\
& x + 2y + S = 2
\end{aligned}$$

$$\text{and} \begin{cases} \lambda S = 0 \\ \mu_1 x = 0 \text{ and } x, y, \lambda, \mu_1, \mu_2 \geq 0 \\ \mu_2 y = 0 \end{cases}$$

To handle last three: either λ or S is nonbasic, μ_1 or x, μ_2 or y

$\Rightarrow$ only one of each pair can be in solution at any time.

QP Conclusions

If we maximize $Z = -A_1 - A_2$, we have solved the original QP. **Why?**

The new problem takes into account the optimization as the 1st-order conditions are met already, plus we have shown the problem to be a maximum as Ω was negative definite.

The advantage of QP is that a QP problem can be re-specified as an LP. Hence, QP problems are treated as separate options/solvers in Matlab and GAMS. In Excel, the problem needs to be set up as an LP as shown above (i.e., solving for the 1st-order conditions).

Price Endogenous Models

Let $P_d = \alpha - \beta\, Q_d$ (demand function)

$P_s = a + b\, Q_s$ (supply function)

In equilibrium:

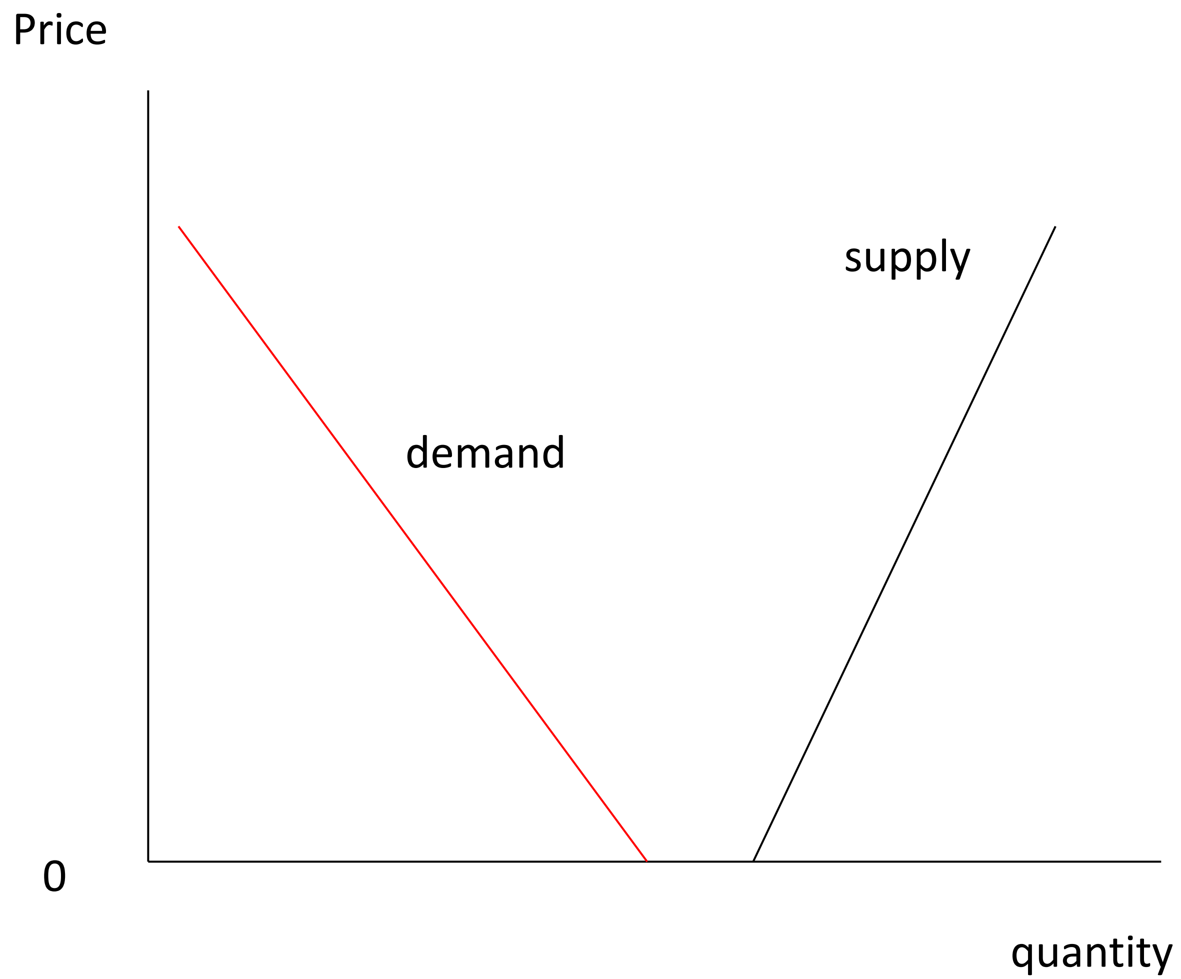
$P_d = P_s$ or $[\alpha - \beta\, Q_d] = [a + b\, Q_s]$

and $Q_d = Q_s$

It is important to recognize that quantity supplied must be equal to or greater than demand $Q_s \geq Q_d$, but if $Q_s > Q_d$, then $P^* = 0$, where P^* is equilibrium price.

Thus: $(-Q_s + Q_d)P^* = 0$

which is a Kuhn-Tucker condition.



Case where supply exceeds demand and price is zero.

Price Endogenous Model (cont)

To solve for the equilibrium quantity and price, the objective is to maximize the area under the demand curve minus the area under the supply function. Thus, we get the following QP problem:

$$\begin{aligned} \text{Max} \quad & \alpha Q_d - \tfrac{1}{2} \beta Q_d^2 - a\, Q_s - \tfrac{1}{2} b\, Q_s^2 \\ \text{s.t.} \quad & Q_d - Q_s \leq 0 \\ & Q_d, Q_s \geq 0 \end{aligned}$$

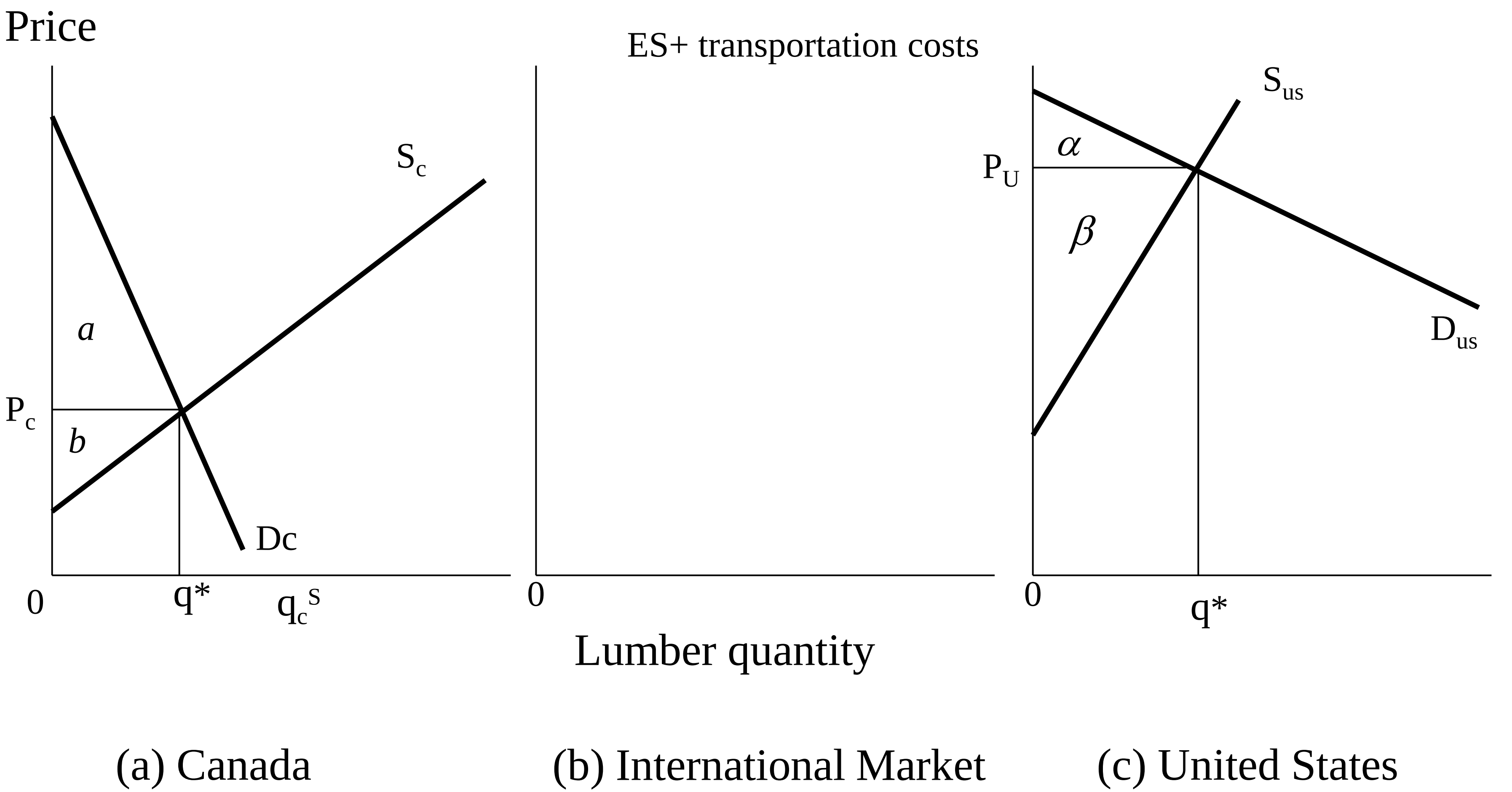
P^* is the **dual variable** associated with the 1st constraint.

Spatial Price Equilibrium (SPE) or Trade Model

- Production and/or consumption occur in spatially separated markets, each with its own supply and demand. Trade occurs if prices between regions differ by the amount of the transportation cost plus tariffs/taxes
- Developed by Takayama & Judge (*Spatial and Temporal Price and Allocation Models* 1971) and Judge & Takayama (*Studies in Economic Planning over Space and Time* 1973) (both North-Holland)

Graphic representation of SPE models follows.

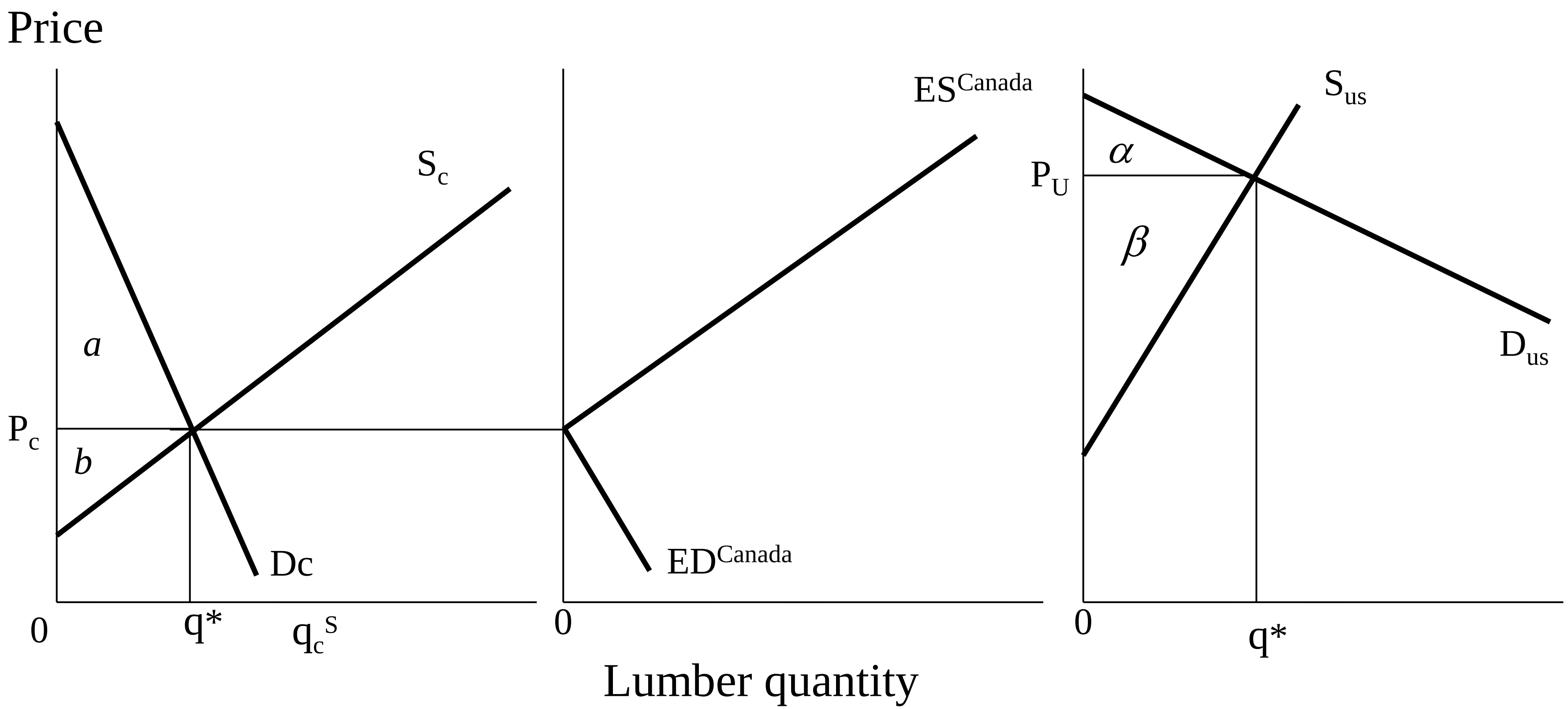
Canada-U.S. Trade in Softwood Lumber



Canadian surplus = $a + b$

U.S. surplus = $\alpha + \beta$

Canada-U.S. Trade in Softwood Lumber (cont)

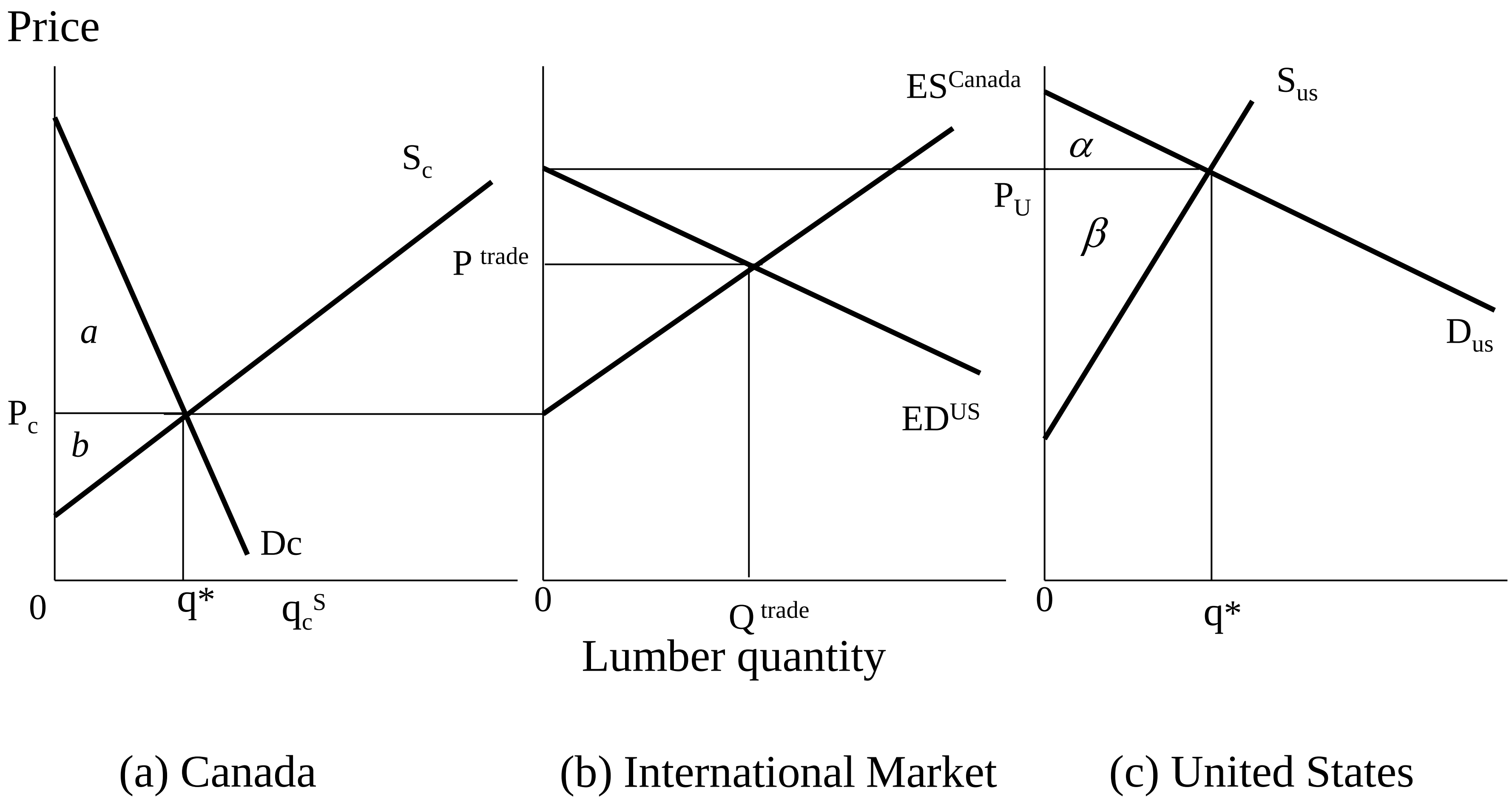


(a) Canada (b) International Market (c) United States

ES = excess supply

ED = excess demand

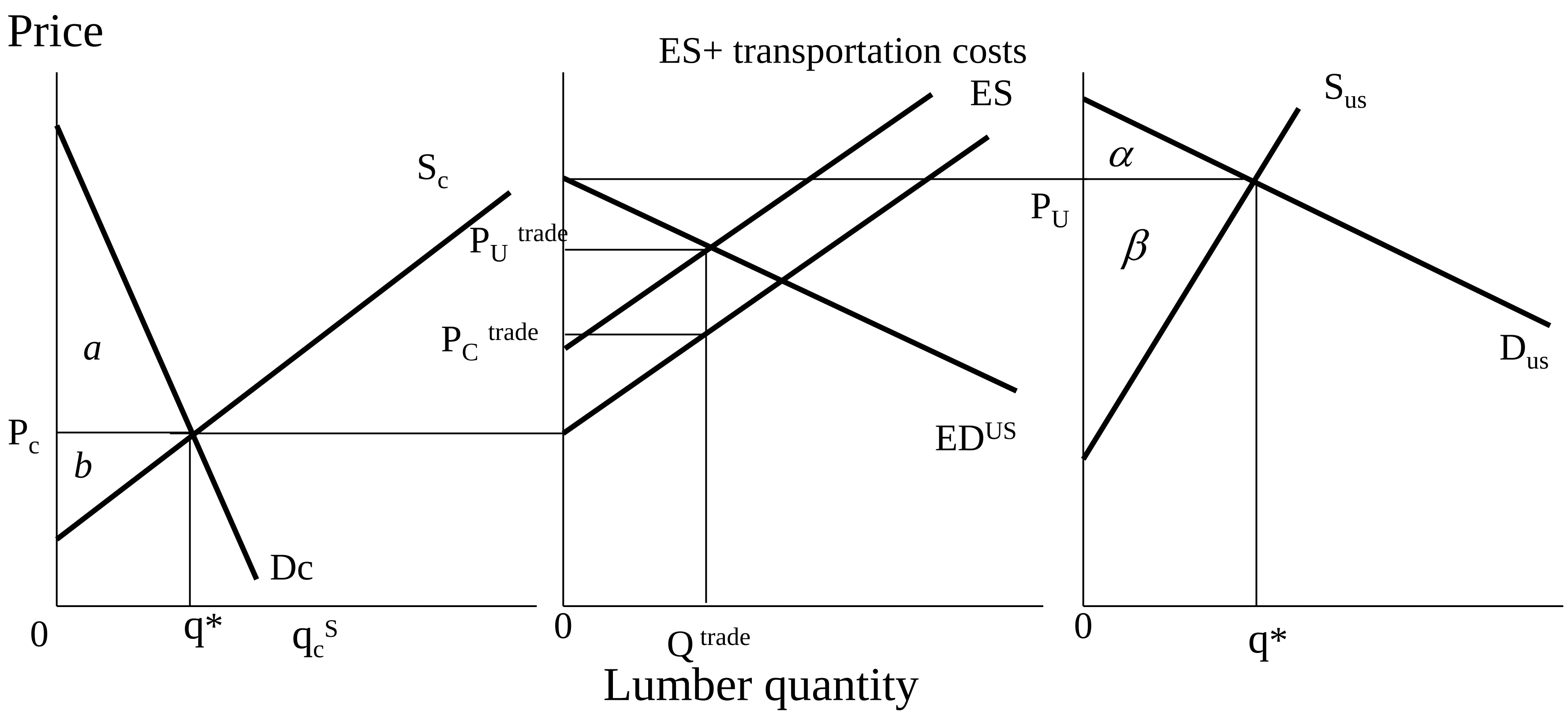
Canada-U.S. Trade in Softwood Lumber (cont)



ES = excess supply

ED = excess demand

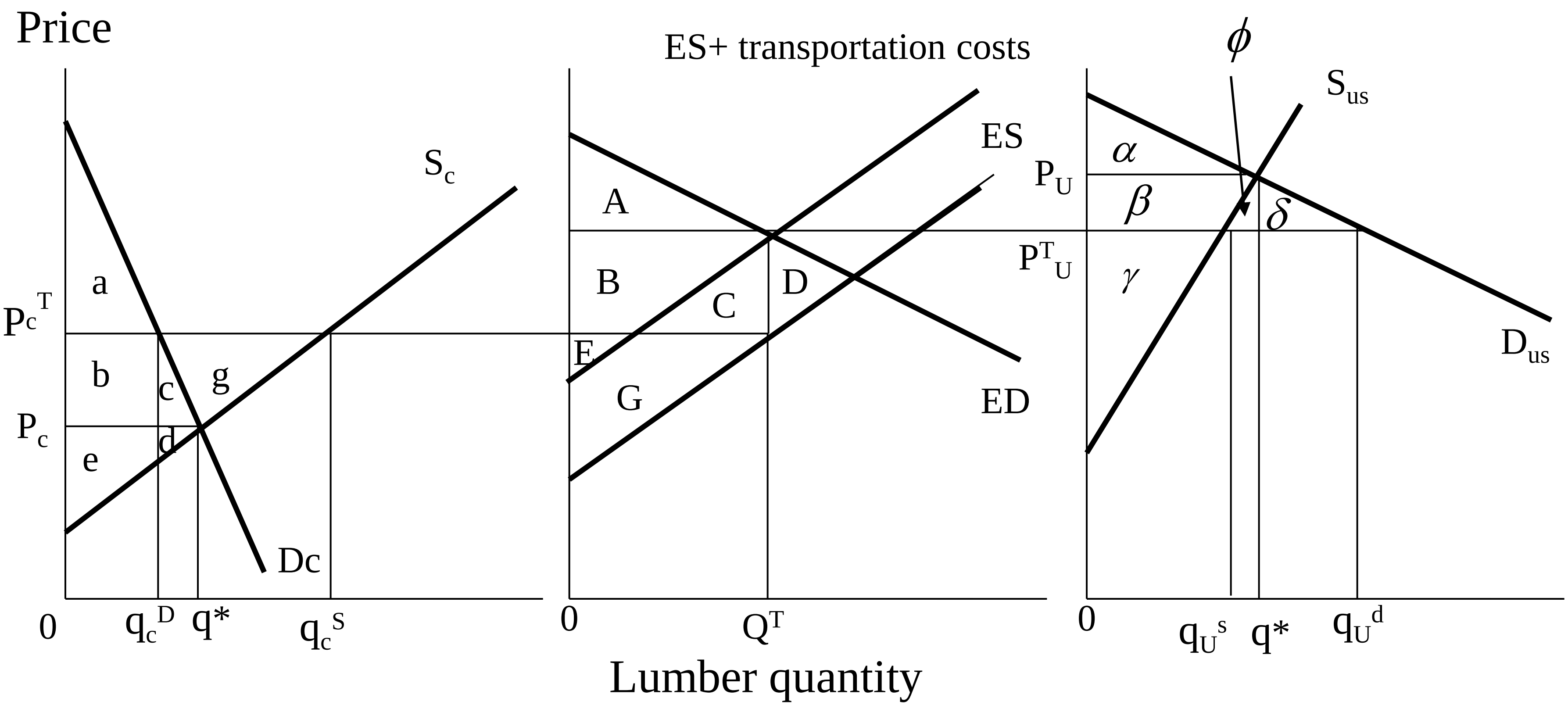
Canada-U.S. Trade in Softwood Lumber (cont)



(a) Canada (b) International Market (c) United States

U.S. and Canadian prices differ by the transportation cost = $P_U^{trade} - P_C^{trade}$

Complete Canada-U.S. Lumber Trade Model



(a) Canada (b) International Market (c) United States

Cdn surplus = $a+b+c+g+e+d$

U.S. surplus = $\alpha+\beta+\phi+\delta+\gamma$

Cdn gain = $g = B+E$

U.S. gain = $\phi+\delta = A$

SPE Model: Mathematical Formulation

$$\max_{q_i, q_j, q_{ij}} Z = \sum_{i=1}^{M} (\alpha_i q_i - 0.5\,\beta_i\, q_i^2)$$

$$- \sum_{j=1}^{X} (a_j q_j + 0.5 b_j q_j^2) - \sum_{i=1}^{M} \sum_{j=1}^{X} t_{ij} q_{ij},\ \forall i, j$$

s.t. $\sum_i q_{ij} \le q_j,\ \forall j$ (region cannot export more than supply)

$\sum_j q_{ij} \ge q_i,\ \forall i$ (regional demands satisfied)

$q_i,\ q_j,\ q_{ij} \ge 0$ (non-negativity)

q_{ij} = sales by region *j* to region *i*,

t_{ij} = unit transport cost from region *j* to region *i*,

X selling regions; *M* buying regions (X≠M, *i* may equal *j*)

Solution Exists IF:

1. Each region's demand is downward sloping
2. Each region's supply is upward sloping
3. Linear demand and supply → quadratic program
4. Z is strictly concave in q_i and q_j, concave in q_{ij}, and bounded from above.

Solution exists and is unique in terms of q_i and q_j, but not necessarily for q_{ij} (see Takayama & Judge p.142)

Example:

Trade between Europe, Japan & U.S. (Ch 13, McCarl & Spreen)

Supplies: $P_{s,U} = 25 + Q_{s,U}$ $\quad P_{s,E} = 35 + Q_{s,E}$ (Only U.S. & Europe supply commodity)

Demands:

$$P_{d,U} = 150 - Q_{d,U}$$
$$P_{d,E} = 155 - Q_{d,E}$$
$$P_{d,J} = 160 - Q_{d,J}$$

Transport costs:

U.S.–Europe = 3 (both directions)

U.S.–Japan = 4

Europe–Japan = 5

GAMS file available here

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******************************************************
* THIS FILE CONTAINS COMMANDS FOR THE EXAMPLE OF *
* SPATIAL EQUILIBRIUM MODEL IN CHAPTER 13         *
******************************************************

$OFFSYMLIST OFFSYMXREF
 OPTION LIMCOL = 0;
 OPTION LIMROW = 0;
* option solprint = off;
SETS  SOURCE    SUPPLY SOURCE LOCATIONS
                /US,EUROPE/
      MARKET   DEMAND MARKETS
               /US,EUROPE,JAPAN/
      THERE(MARKET,SOURCE) SET WHICH MATCHES MARKETS
              /US.US,EUROPE.EUROPE/
      CURVE    CURVE PARAMETERS   /INTERCEPT,SLOPE/

TABLE   SUPPLYEQ(SOURCE,CURVE)  INVERSE SUPPLY CURVES
                INTERCEPT    SLOPE
      US            25          1
      EUROPE        35          1

TABLE   DEMANDEQ(MARKET,CURVE)  INVERSE DEMAND CURVES
                INTERCEPT    SLOPE
      US           150         -1
      EUROPE       155         -1
      JAPAN        160         -1

TABLE   COST(SOURCE,MARKET)   SHIPPING COST FROM EACH SOURCE TO EACH MARKET

                     US      EUROPE    JAPAN
         US           0        3        4
         EUROPE       3        0        5


PARAMETERS
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PARAMETERS
       LIMITS(SOURCE) QUOTAS ON OUTGOING SHIPMENTS
       SUBTAX(SOURCE) SUBSIDIES AND OR TAXES ON OUTGOING SHIPMENTS;
       LIMITS(SOURCE) = 99999.;
       SUBTAX(SOURCE) = 0.;

POSITIVE VARIABLES
         SHIPMENTS(SOURCE,MARKET) AMOUNT SHIPPED OVER A TRANPORT ROUTE
         SUPPLY(SOURCE)           QUANTITY AVAILABLE AT EACH SOURCE
         DEMAND(MARKET)           QUANTITY REQUIRED BY DEMAND MARKET

VARIABLES
         CSPS                   TOTAL CONSUMERS AND PRODUCERS SURPLUS;

EQUATIONS
         TSURP                   TOTAL SURPLUS EQUATION
         SUPPLYE(SOURCE)         LIMIT ON SUPPLY AVAILABLE AT A SOURCE
         DEMANDE(MARKET)         MINIMUM REQUIREMENT AT A DEMAND MARKET
         QUOTA(SOURCE)           OUTGOING SHIPMENT QUOTA BY SOURCE;

TSURP..  CSPS =E=
   SUM(MARKET,DEMANDEQ(MARKET,"INTERCEPT")*DEMAND(MARKET)
       +0.5*DEMANDEQ(MARKET,"SLOPE")*DEMAND(MARKET)*DEMAND(MARKET))
   - SUM(SOURCE,SUPPLYEQ(SOURCE,"INTERCEPT")*SUPPLY(SOURCE)
       +0.5*SUPPLYEQ(SOURCE,"SLOPE")*SUPPLY(SOURCE)*SUPPLY(SOURCE))
   - SUM((SOURCE,MARKET),SHIPMENTS(SOURCE,MARKET)
       *(COST(SOURCE,MARKET)+SUBTAX(SOURCE)$(NOT THERE(MARKET,SOURCE)))) ;

DEMANDE(MARKET)..   DEMAND(MARKET)
                     - SUM(SOURCE,SHIPMENTS(SOURCE,MARKET)) =L= 0;

SUPPLYE(SOURCE)..   -SUPPLY(SOURCE)
                     + SUM(MARKET,SHIPMENTS(SOURCE,MARKET)) =L= 0;
```

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```
EQUATIONS
        TSURP                  TOTAL SURPLUS EQUATION
        SUPPLYE(SOURCE)        LIMIT ON SUPPLY AVAILABLE AT A SOURCE
        DEMANDE(MARKET)        MINIMUM REQUIREMENT AT A DEMAND MARKET
        QUOTA(SOURCE)          OUTGOING SHIPMENT QUOTA BY SOURCE;

TSURP..  CSPS =E=
  SUM(MARKET,DEMANDEQ(MARKET,"INTERCEPT")*DEMAND(MARKET)
      +0.5*DEMANDEQ(MARKET,"SLOPE")*DEMAND(MARKET)*DEMAND(MARKET))
  - SUM(SOURCE,SUPPLYEQ(SOURCE,"INTERCEPT")*SUPPLY(SOURCE)
      +0.5*SUPPLYEQ(SOURCE,"SLOPE")*SUPPLY(SOURCE)*SUPPLY(SOURCE))
  - SUM((SOURCE,MARKET),SHIPMENTS(SOURCE,MARKET)
      *(COST(SOURCE,MARKET)+SUBTAX(SOURCE)$(NOT THERE(MARKET,SOURCE)))) ;

DEMANDE(MARKET)..  DEMAND(MARKET)
                    - SUM(SOURCE,SHIPMENTS(SOURCE,MARKET)) =L= 0;

SUPPLYE(SOURCE)..  -SUPPLY(SOURCE)
                    + SUM(MARKET,SHIPMENTS(SOURCE,MARKET)) =L= 0;

QUOTA(SOURCE)..    SUM(MARKET$(NOT THERE(MARKET,SOURCE))
                       ,SHIPMENTS(SOURCE,MARKET)) =L= LIMITS(SOURCE);

MODEL TRANSPORT /ALL/;

SOLVE TRANSPORT USING NLP MAXIMIZING CSPS;

 parameter report(*,*,*);
 parameter trans(source,market);
          report(market,"demand","quantity") = demand.l(market);
          report(source,"supply","quantity") = supply.l(source);
          report(market,"demand","price") = demande.m(market);
          report(source,"supply","price") = supplye.m(source);
          trans(source,market) = shipments.l(source,market);
 display report,trans;
```

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          report(source,"supply","price") = supplye.m(source);
          trans(source,market) = shipments.l(source,market);
display report,trans;

limits(source)=0;
SOLVE TRANSPORT USING NLP MAXIMIZING CSPS;
          report(market,"demand","quantity") = demand.l(market);
          report(source,"supply","quantity") = supply.l(source);
          report(market,"demand","price") = demande.m(market);
          report(source,"supply","price") = supplye.m(source);
          trans(source,market) = shipments.l(source,market);
display report,trans;

limits(source)=99999;
limits("us") = 2;
SOLVE TRANSPORT USING NLP MAXIMIZING CSPS;
          report(market,"demand","quantity") = demand.l(market);
          report(source,"supply","quantity") = supply.l(source);
          report(market,"demand","price") = demande.m(market);
          report(source,"supply","price") = supplye.m(source);
          trans(source,market) = shipments.l(source,market);
display report,trans;

limits(source)=99999;
subtax("europe")=-1.;
subtax("us")=1.;
SOLVE TRANSPORT USING NLP MAXIMIZING CSPS;
          report(market,"demand","quantity") = demand.l(market);
          report(source,"supply","quantity") = supply.l(source);
          report(market,"demand","price") = demande.m(market);
          report(source,"supply","price") = supplye.m(source);
          trans(source,market) = shipments.l(source,market);
display report,trans;
```

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	quantity	shadow price
US .demand	46.400	103.600
US .supply	78.600	103.600
EUROPE.demand	50.400	104.600
EUROPE.supply	69.600	104.600
JAPAN .demand	51.400	108.600

Sales to → from	US	EUROPE	JAPAN
US	46.40	0	32.20
EUROPE	0	50.40	19.20

Objective value = 9193.60

Reduced cost: US to Europe = -4
Europe to US = -2
All others = 0

Scenario Analysis

	Undistorted	No-Trade	Quota	Tax/Subsidy
Objective	9193.6	7506.3	8761.6	9178.6
U.S. Demand	45.4	62.5	61.5	46.4
U.S. Supply	9.6	62.5	63.5	78.6
U.S. Price	104.6	87.5	88.5	103.6
Europe Demand	51.4	60	40.7	50.4
Europe Supply	68.6	60	79.3	69.6
Europe Price	103.6	95	114.3	104.6
Japan Demand	51.4	0	40.7	51.4
Japan Price	108.6	160	119.3	108.6