Resource Economics: Introduction

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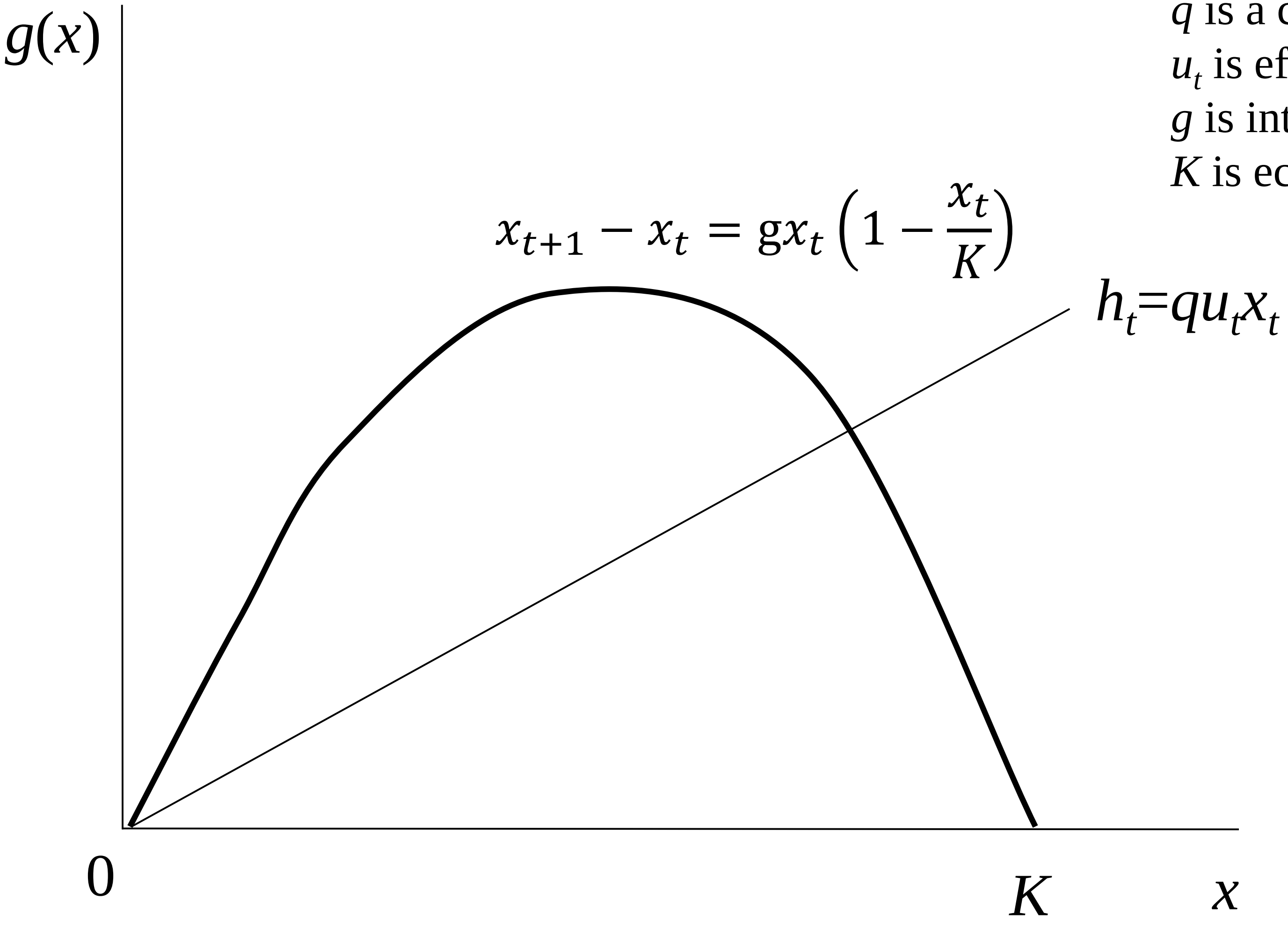
Logistic growth function: $g(x_t)$

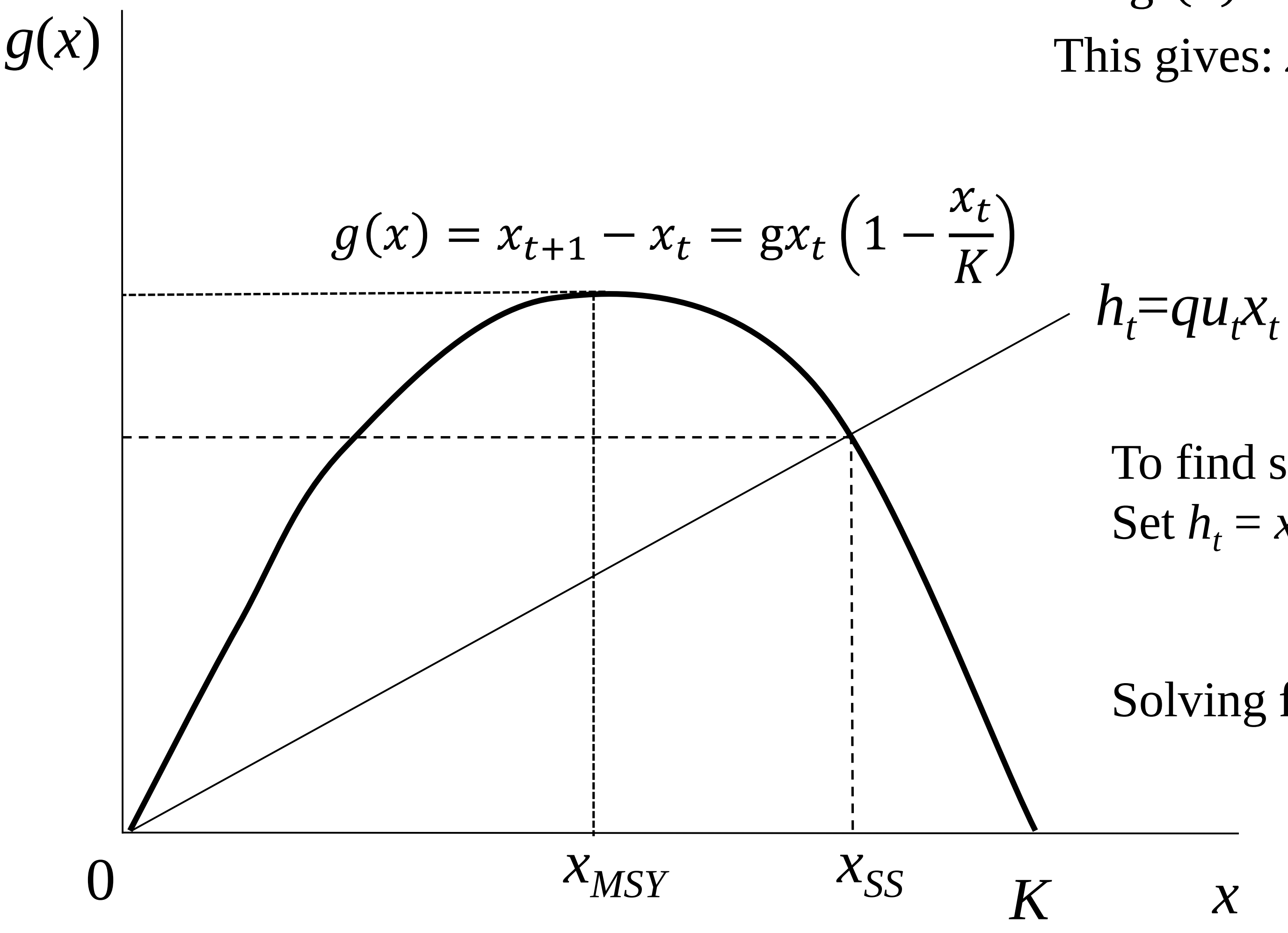
q is a catchability coefficient
u_t is effort (control variable)
g is intrinsic growth rate
K is ecosystem carrying capacity



To find maximum sustainable yield:

Set $\mathrm{g}'(x) = \mathrm{g} - \frac{2gx}{K} = 0$

This gives: $x_{MSY} = ½\, K$



To find steady state solution:

Set $h_t = x_{t+1} - x_t$

$$\mathrm{g}\left(1 - \frac{x_t}{K}\right) = qu_t$$

Solving for x_t gives: $x_{SS} = \frac{K(\mathrm{g} - qu)}{\mathrm{g}}$

Non-linearity is problematic.

The *logistics growth function* (usually specified as a constraint in bioeconomics):

$$x_{t+1} = g\, x_t\, (1 - x_t)$$

where $0 \leq x \leq 1$ is the proportion of a wildlife population or fish biomass, and g is the intrinsic growth rate

x=0 → extinction

x=1 → carrying capacity of system

Choose g = 2.7 and x_0 = 0.5 → convergence to x=0.6296

As g increases to 2.9, 3.0, 3.5, 3.82, 3.83, 3.84, 3.85 one gets bifurcations and chaos

Logistics growth function (cont)

$$x_{t+1} - x_t = \mathrm{g}x_t\left(1 - \frac{x_t}{K}\right)$$

- $0 < g \leq 1$: steadily approach K without overshooting
- $1 < g \leq 2$: overshoots K but approaches K with damped oscillations
- $2 < g \leq 2.449$: two-point cycles about K (bifurcation)
- $2.449 < g \leq 2.570$: stable cycles with 2^n points, $n \geq 1$ (n depends on g)
- $g \geq 2.570$: irregular, non-periodic behaviour that leads to dynamic chaos

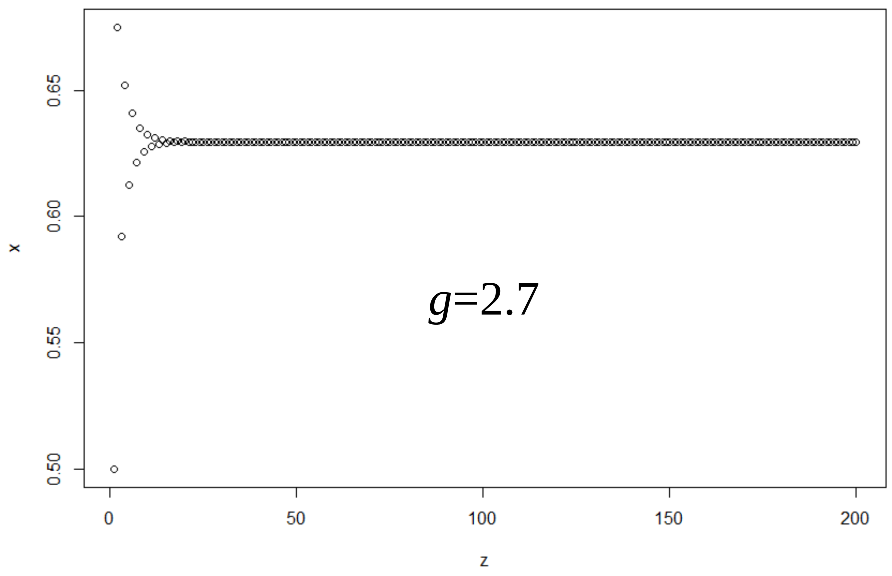
MATLAB code for logistics growth function: bifurcation and chaos

(Can also do this in Octave, which is free software.)

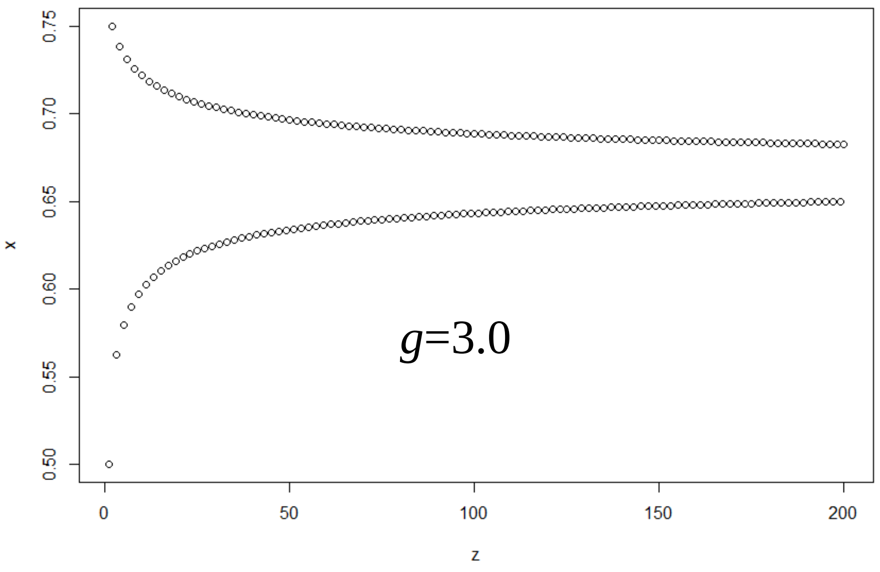
```
% Logistics bifurcation diagram

clear all
N = 1000;
x = zeros(1,N+1);
x(1) = 0.2;
% r = [2.7 2.9 2.95 2.99 3 3.1 3.5];
g = 2.7:.001:4;
figure;
for i=1:length(r)
% R = g(i)
for t=1:N
x(t+1) = g(i)*x(t)*(1-x(t));
end
% X = x(N+1)
hold on
plot(g(i)*ones(1,100),x(N-
98:N+1),'.','LineStyle','none','MarkerSize',.1,'color', 'k')
% figure;
% plot(x,'.','LineStyle','none','MarkerSize',.1,'color', 'k')
% xlim([0 1000])
% ylim([0 1])
% xlabel('# of iterations')
% ylabel('x')
% title([‘g = ' num2str(r(i))])
end
xlabel(‘g')
ylabel('x')
```

Simple R code for logistics growth function: bifurcation and chaos

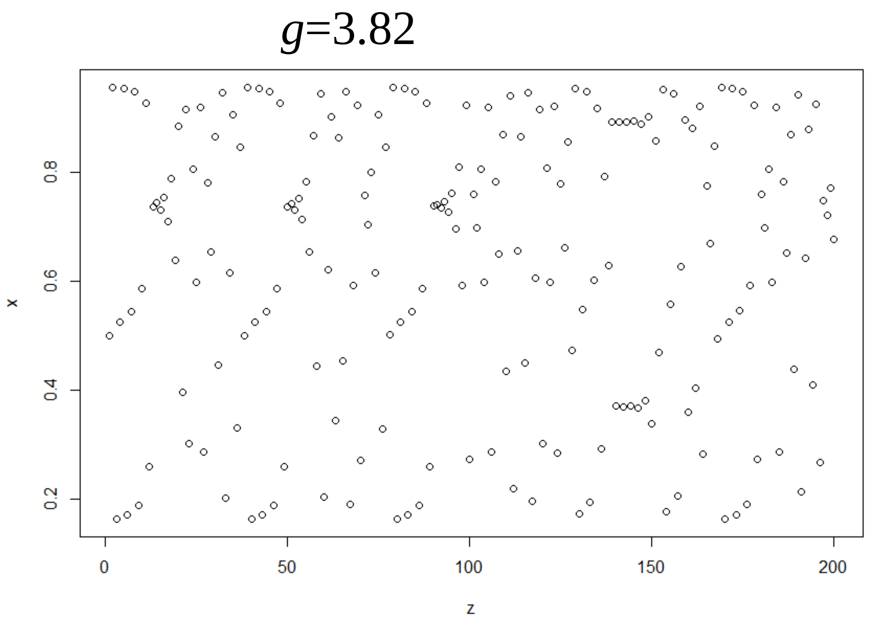
```
# Logistics bifurcation diagram

N <- 199   #Number of iterations
K <- N+1
x <- rep(0,N)
x[1] <- 0.5 # Starting value for fish stock
g <- 2.7  # values of intrinsic growth rate: 2.7, 2.9, 3.0, 3.5, 3.82, 3.83, 3.84, 3.85
z <- c(1:K)

for (t in 1:N) {
   x[t+1] = g*x[t]*(1-x[t])
}

plot(z,x)
```



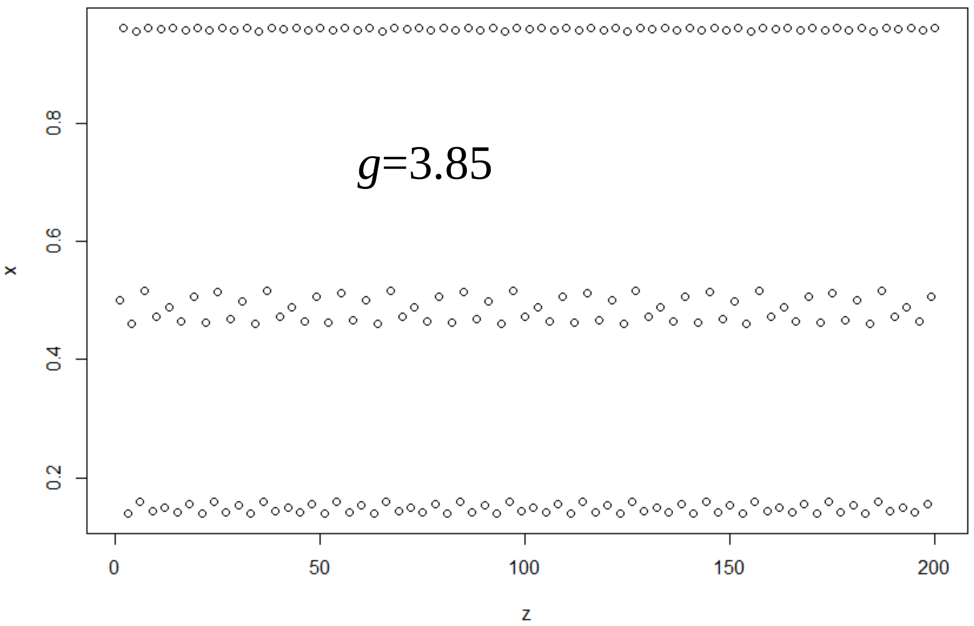
0.50
0.55
0.60
0.65
x
g=2.7
0
50
100
150
200
z



0.50
0.55
0.60
0.65
0.70
0.75
x
g=3.0
0
50
100
150
200
z



g=3.82
0.2
0.4
0.6
0.8
x
0
50
100
150
200
z



0.2
0.4
0.6
0.8
x
g=3.85
0
50
100
150
200
z

Logistics growth and non-linearity (cont.).

The *logistics growth function* can also be specified as follows:

$$x_{t+1} = x_t + g\, x_t\, (1 - x_t)$$

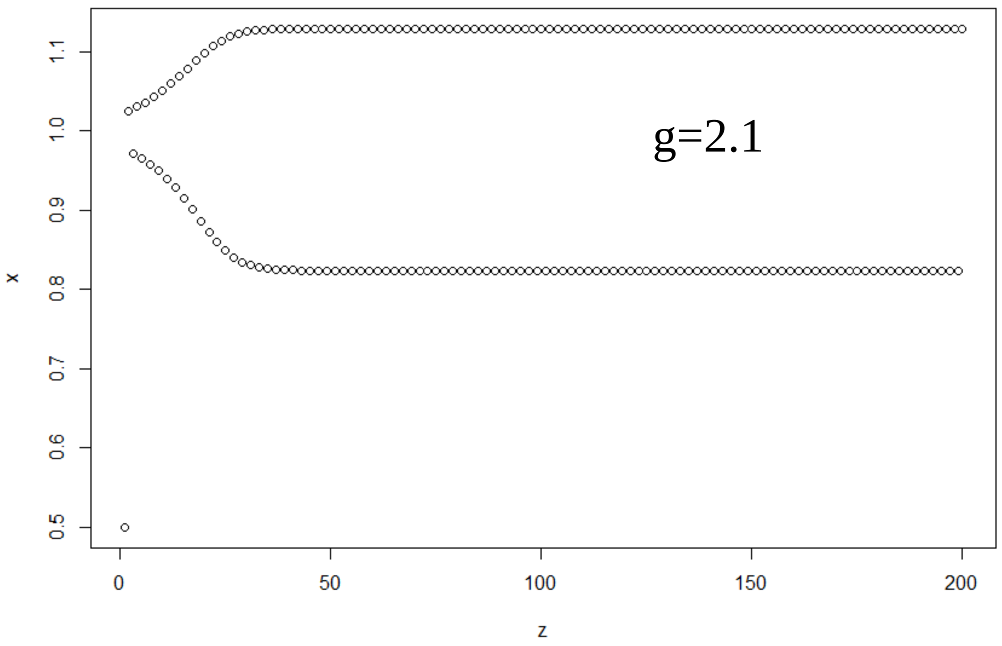
where $0 \le x \le 1$ is the proportion of a wildlife population or fish biomass, and g is the intrinsic growth rate

$x=0 \rightarrow$ extinction

$x=1 \rightarrow$ carrying capacity of system

Now the use of $g = 1.7$ and $x_0 = 0.5 \rightarrow$ convergence to $x=1.0$

As g increases we again get bifurcations ($g=2.1$) and chaos ($g=2.7$)


g=2.1
x
z
0.5
0.6
0.7
0.8
0.9
1.0
1.1
0
50
100
150
200

Discounting

Suppose you receive $\$x$ from now into perpetuity. What is that worth?
Further suppose the discount rate is δ $(0 < \delta \leq 1)$. Let:

$$(1) \qquad V_n = x + \frac{x}{1+\delta} + \frac{x}{(1+\delta)^2} + \cdots + \frac{x}{(1+\delta)^n}$$

$$(2) \qquad \left(\frac{1}{1+\delta}\right) V_n = \frac{x}{1+\delta} + \frac{x}{(1+\delta)^2} + \cdots + \frac{x}{(1+\delta)^n} + \frac{x}{(1+\delta)^n}$$

Subtract (2) from (1):

$$(3) \qquad V_n - \left(\frac{1}{1+\delta}\right) V_n = x - \frac{x}{(1+\delta)^{n+1}} = x\left[1 - \frac{1}{(1+\delta)^{n+1}}\right]$$

$$(4) \qquad \left(\frac{\delta}{1+\delta}\right) V_n = x\left[1 - \frac{1}{(1+\delta)^{n+1}}\right]$$

$$(5) \qquad \mathrm{V} = \lim_{n\to\infty} V_n = \lim_{n\to\infty} \frac{x(1+\delta)}{\delta}\left[1 - \cancelto{0}{\frac{1}{(1+\delta)^{n+1}}}\right]$$

Thus, $V = x\left(\frac{1+\delta}{\delta}\right)$

Discounting #2

Suppose instead you begin to receive $\$x$ one period from now and then into perpetuity. What is that worth? Let:

(1) $$V_n = \frac{x}{1+\delta} + \frac{x}{(1+\delta)^2} + \cdots + \frac{x}{(1+\delta)^n}$$

(2) $$\left(\frac{1}{1+\delta}\right) V_n = \frac{x}{(1+\delta)^2} + \frac{x}{(1+\delta)^3} + \cdots + \frac{x}{(1+\delta)^n} + \frac{x}{(1+\delta)^n}$$

Subtract (2) from (1):

(3) $$V_n - \left(\frac{1}{1+\delta}\right) V_n = \frac{x}{1+\delta} - \frac{x}{(1+\delta)^{n+1}} = \frac{x}{1+\delta}\left[1 - \frac{1}{(1+\delta)^n}\right]$$

(4) $$\left(\frac{\delta}{1+\delta}\right) V_n = \frac{x}{1+\delta}\left[1 - \frac{1}{(1+\delta)^n}\right]$$

(5) $$\mathrm{V} = \lim_{n\to\infty} V_n = \lim_{n\to\infty} \frac{x}{\delta}\left[1 - \cancelto{0}{\frac{1}{(1+\delta)^n}}\right]$$

Thus, $V = \frac{x}{\delta}$

Problem of the mine

Suppose you need to determine the optimal production schedule $\{y_t\}$ for removing ore from a mine that is to be shut down and abandoned at time t=10. Price of ore is p=1and the extraction cost is $c_t = y_t^2/x_t$. Here x_t refers to the reserves remaining at time t. Assume initial reserves of x_0 = 1,000. What does the problem look like?

$$\max_{y_t} \pi_t = p\, y_t - \frac{y_t^2}{x_t} = \left(1 - \frac{y_t}{x_t}\right) y_t$$
$$s.t. \quad x_{t+1} - x_t = -y_t$$

This problem assumes no discounting. The Lagrangian for the problem can be written as:

$$L = \sum_{t=1}^{10} \left[\left(1 - \frac{y_t}{x_t}\right) y_t + \lambda_{t+1}(-y_t - x_{t+1} + x_t) \right]$$

It is much simpler to write the Hamiltonian as:

$$H = \left(1 - \frac{y_t}{x_t}\right) y_t - \lambda_{t+1} y_t$$

The first-order conditions are:

$$(1)\quad \frac{\partial H}{\partial y_t} = 0 \Rightarrow 1 - \frac{2y_t}{x_t} - \lambda_{t+1} = 0$$

$$(2)\quad \lambda_{t+1} - \lambda_t = -\frac{\partial H}{\partial x_t} \Rightarrow \lambda_{t+1} - \lambda_t = \frac{y_t^2}{x_t^2}$$

$$(3)\quad x_{t+1} - x_t = \frac{\partial H}{\partial \lambda_{t+1}} \Rightarrow x_{t+1} - x_t = -y_t$$

$$x_0 = 0 \text{ and } \lambda_{10} = S'(x_{10}) = 0$$

What does the value of λ_{10} tell us?

Mine problem (cont.)

This is a problem with 31 equations in 31 unknowns: y_t for $t = 0, 1, \ldots, 9$; x_t for $t = 0, 1, \ldots, 10$; and λ_t for $t = 1, \ldots, 10$
You can solve this problem in GAMS, say, or in Excel using the solver
You can also solve it by defining $z_t = y_t/x_t$.

From (1) on the previous slide: $1 - 2z_t = \lambda_{t+1}$. Given $\lambda_{10} = 0$, $z_9 = ½$.
Then, from (2) $\lambda_t = \lambda_{t+1} + z_t$ implies $\lambda_9 = \lambda_{10} + z_9 = ½$.
In this way we can find all of the values for λ and z.

Finally, we can unravel the values of x and y using $y_t = x_t \times z_t$:
$x_0 = 1{,}000$, so $y_0 = 1000 \times z_0$. Since we found $z_0 = 0.1389$, then $y_0 = 138.9$.
Then, using equation (3), $x_{t+1} = x_t - y_t$, so $x_1 = x_0 - y_0 = 1000 - 138.9 = 861.1$

Now write a program in R to solve this problem.

The fishery problem can be specified as follows:

$$\text{Max} \quad Z = \sum_{t=0}^{T-1} \beta^t R(x_t, u_t) + \beta^T S(x_T)$$

$$\text{s.t.} \qquad x_{t+1} - x_t = g x_t \left(1 - \frac{x_t}{K}\right) - h(u_t)$$

$$x_0 = \bar{x} \text{ (given)}$$

- x_t is the stock of fish at time *t* (measured as a mass, say in kg or tons)
- $\beta = 1/(1+r)$ with *r* the discount rate is the discount factor
- Growth is modeled as a logistics function
- $h(u_t)$ is harvest as a function of effort *u* at time t

Re-write the growth equation (dynamic constraint or equation of motion) as:

$$x_{t+1} = x_t - h(u_t) + gx_t\left(1 - \frac{x_t}{K}\right)$$

We can refer to $x_t - h(u_t)$ as *escapement*.

Escapement plus net growth gives the mass of fish available in period t+1.

The lagrangian for the above problem can be written as follows:

$$L = \sum_{t=0}^{T-1} \beta^t R(x_t, u_t) + \beta^T S(x_T) + \lambda_0(\bar{x} - x_0)$$
$$+ \sum_{t=0}^{T-1} \beta^{t+1} \lambda_{t+1} \left[g x_t \left(1 - \frac{x_t}{K} \right) - x_{t+1} + x_t \right]$$

How to solve this problem analytically is the subject of a discrete dynamic optimization video: http://web.uvic.ca/~kooten/video/DiscreteDynamicOptimization.mp4
(A continuous time version: http://web.uvic.ca/~kooten/video/ContinuousDynamicOptimization.mp4)

We can solve the problem numerically as a constrained optimization (a nonlinear mathematical programming) problem using GAMS if we are provided values of the parameters *g*, *K*, *β* (or *r*), *T*, $\bar{x}$ and the range of controls available to the decision maker.

Another way to write the Lagrangian is as follows:

$$L=\sum_{t=0}^{T-1}\left\{\beta^{t}\left[R(x_t,u_t)+\beta\lambda_{t+1}\left(gx_t\left(1-\frac{x_t}{K}\right)-x_{t+1}+x_t\right)\right]\right\}+\lambda_0(\bar{x}-x_0)+\beta^{T}S(x_T)$$

Here it is clear that the term $\beta\lambda_{t+1}$ refers to the present value of the shadow price (marginal cost of leaving the fish unharvested. The term $\beta\lambda_{t+1}$ is also known as the *user cost* or discounted shadow price of remaining reserves in period t+1.

Discrete-time, current-value Hamiltonian
(*h* is the control variable harvest and *f*(*x*) is the growth equation)

$$\max \quad \sum_{t=0}^{\infty} \beta^t R(x_t, h_t)$$

$$\text{s.t.} \quad x_{t+1} - x_t = f(x_t) - h_t$$

$$x_0 = \bar{x}_0 \ \text{ given}$$

Lagrangian is given by

$$\mathrm{L} = \sum_{t=0}^{\infty} \beta^t \left\{ R(x_t, h_t) - \beta\lambda_{t+1}\left[x_{t+1} - x_t - f(x_t) + h_t \right] \right\}$$

Also the transversality condition

$$\lim_{n\to\infty} \beta^t \lambda_t x_t = 0$$

is assumed to hold.

Current-value Hamiltonian

$$H_t = \quad R(x_t, h_t) + \beta\lambda_{t+1}\left[f(x_t) - h_t\right]$$

Lagrangian is given by

$$\mathrm{L} = \sum_{t=0}^{\infty} \beta^t \left[H_t + \lambda_{t+1}(x_t - x_{t+1})\right]$$

Weitzman refers to the optimized Hamiltonian as the *properly accounted income* because the 1st term on the RHS is the *dividend* from optimal management of the resource (fish stock) while the 2nd term is the *capital gain* from the optimal management of the resource today and in the future. Together then they are the properly accounted income.

To find the optimal levels of x_t and h_t, however, we first solve for the first-order conditions (FOC). This is shown on the next slide.

1st - order conditions:

$$\frac{\partial H}{\partial h_t} = 0 \Rightarrow \frac{\partial R}{\partial h_t} = \beta\lambda_{t+1}$$

$$\beta\lambda_{t+1} - \lambda_t = -\frac{\partial H}{\partial x_t} \Rightarrow \lambda_t = \frac{\partial R}{\partial x_t} + \beta\lambda_{t+1}\left(1 + \frac{\partial f}{\partial x_t}\right)$$

$$x_{t+1} - x_t = \frac{\partial L}{\partial(\beta\lambda_{t+1})} \Rightarrow x_{t+1} = x_t + f(x_t) - h_t$$

Dynamic Programming

The *value function* of dynamic programming:

$$V(x_t) \equiv \underset{\{h_s\}}{\text{maximize}} \left[\sum_{s=t}^{\infty} \beta^{s-t} R(x_s, h_s) \right] \equiv \sum_{s=t}^{\infty} \beta^{s-t} R(x_s^*, h_s^*)$$

The forgoing procedure leads to the following recursive equation, known as **Bellman's Equation**:

$$V_t(x_t, h_t) = \max_{h_t} \left[R(x_t, h_t) + \beta V_{t+1}(x_{t+1}) \right]$$

where $x_{t+1} = x_t + f(x_t) - h_t$

Dynamic Programming (cont):

$$1^{\text{st}} - \text{order conditions}:$$

$$(1)\ \frac{\partial V_t}{\partial x_t} = \frac{\partial R}{\partial x_t} + \beta \frac{dV_{t+1}}{dx_{t+1}} \cdot \frac{\partial x_{t+1}}{\partial x_t}$$

$$(2)\ \frac{\partial V_t}{\partial h_t} = \frac{\partial R}{\partial h_t} + \beta \frac{dV_{t+1}}{dx_{t+1}} \cdot \frac{\partial x_{t+1}}{\partial h_t} = 0$$

Consider equation (2): $\frac{\partial R}{\partial h_t} + \beta \frac{dV_{t+1}}{dx_{t+1}} \cdot (-1) = 0 \ \Rightarrow \ \frac{\partial R}{\partial h_t} = \beta \frac{dV_{t+1}}{dx_{t+1}} = \beta \lambda_{t+1}$

Recall result from Lagrange method above: $\frac{\partial R}{\partial h_t} = \beta \lambda_{t+1}$

Notice $dV_{t+1}/dx_{t+1} = V'(x_{t+1})$ is the marginal value of an increase in x_{t+1}, which is simply λ_{t+1}. Then, we need to discount the marginal value to the present time, so:

$$\beta\, V'(x_{t+1}) = \beta\ \ \lambda_{t+1}. \quad \text{(equation 2)}$$

We can also show that equation (1) leads to the co-state equation as with the Hamiltonian and Lagrangian methods.

Rewrite (1) as:

$$\lambda_t = \frac{\partial R}{\partial x_t} + \beta\lambda_{t+1}(1 + \frac{\partial f}{\partial x_t}) \Rightarrow \lambda_t = \frac{\partial R}{\partial x_t} + \beta\lambda_{t+1} + \beta\lambda_{t+1}\frac{\partial f}{\partial x_t}$$

$$\Rightarrow \beta\lambda_{t+1} - \lambda_t = -\left[\frac{\partial R}{\partial x_t} + \beta\lambda_{t+1}\frac{\partial f}{\partial x_t}\right]$$

Rewrite (2) as:

$$\frac{\partial R}{\partial h_t} + \beta\lambda_{t+1}\frac{\partial f}{\partial h_t} = 0$$

Note: The above two results are exactly the same as those one gets from discrete-time Lagrangian and from the discrete-time Hamiltonian. Further, in a competitive, arbitrage-free economy (where *r* is the discount rate):

$$rV(x_t) = H^*{}_t$$

Schaefer Model of the Fishery (1954)

$$(1) \quad G(x) = \mathrm{g}x\left(1 - \frac{x}{K}\right)$$

$$(2) \quad h(E, x) = qEx$$

Equation (1) is the growth of the fish stock in the absence of harvest and (2) is the production function (with E = effort). Assume that the equation of motion is given by:

$$(3) \quad \frac{\Delta x}{\Delta t} = G(x) - h$$

From (3) with (1) and (2), and assuming steady state dynamics, we get:

$$h = rx(1 - x/K) = qEx$$

Solving for x gives:

$$(4) \qquad x = K(1 - qE/r)$$

so that

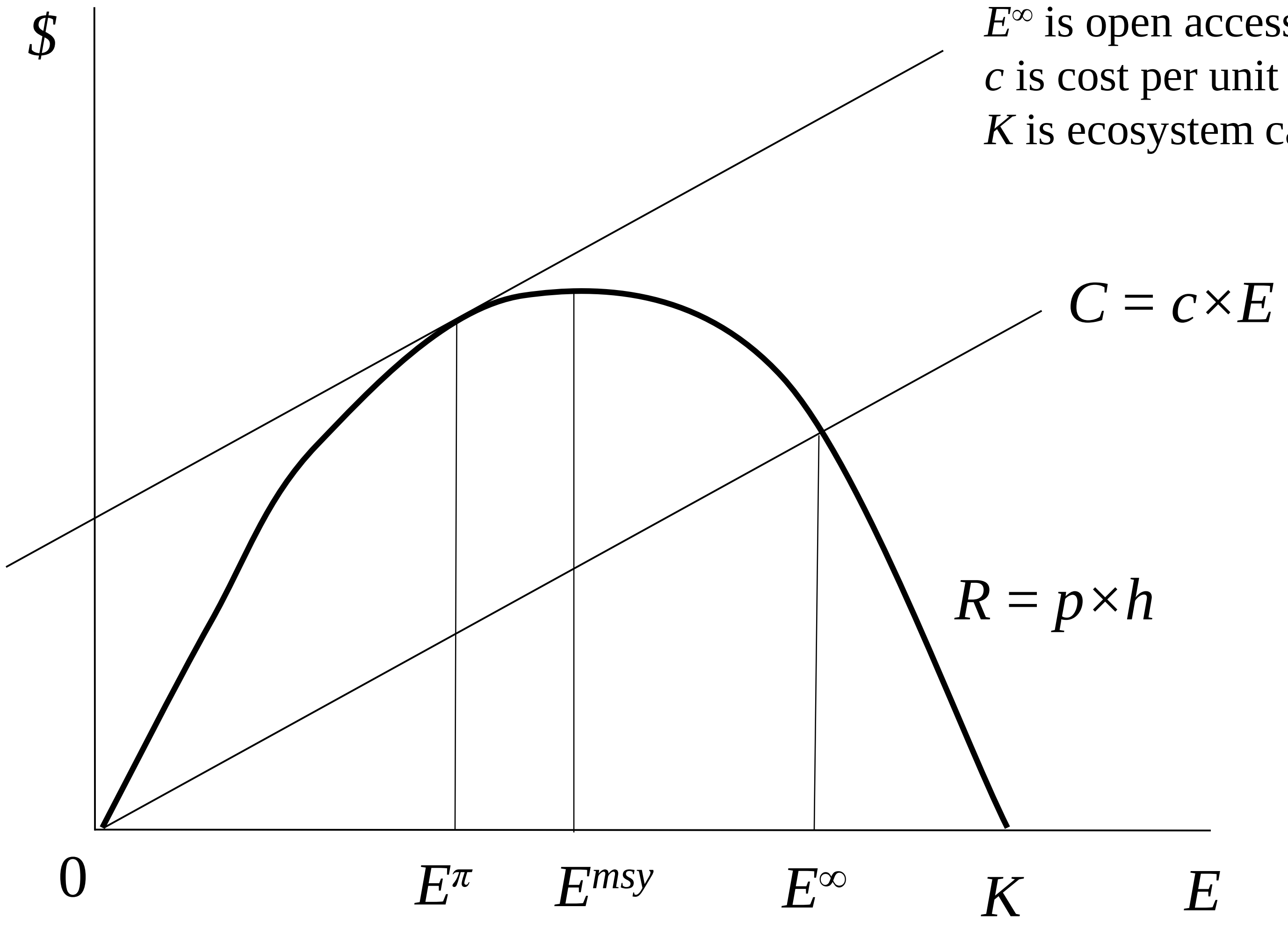
$$(5) \qquad h = qEx = qKE(1 - qE/r)$$

Schaefer Model of the Fishery (cont)

- Equation (5) has exactly the same form as the logistics growth function that we encountered previously, except E (effort) is on the horizontal axis and h (harvest) is on the vertical axis.
- Now, if you multiply this by a fixed price for fish, then you essentially get a total revenue function that looks like a logistics.

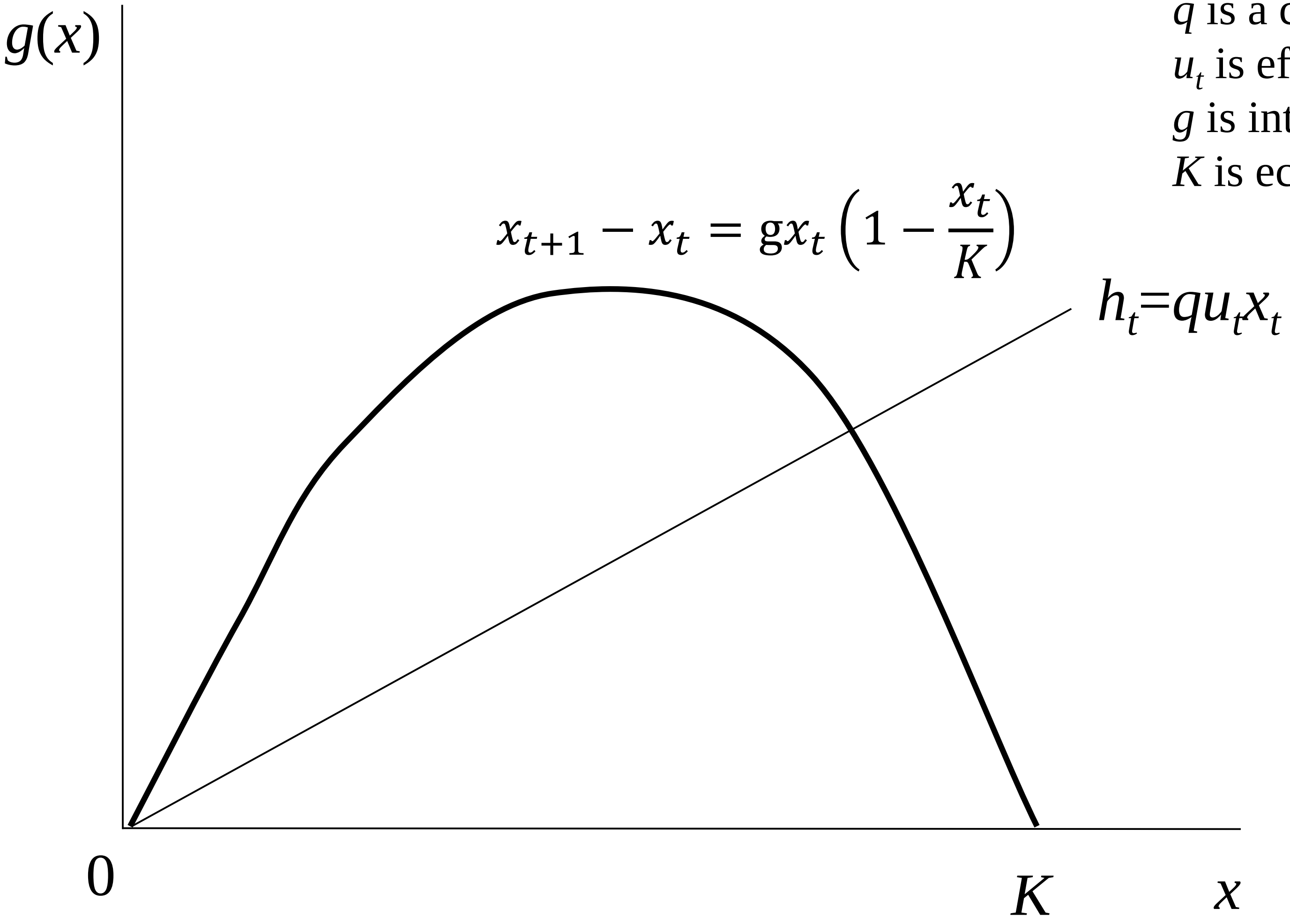
Fishery model:

p is price of fish ($/kg)
h is harvest
E is effort
E^{∞} is open access effort
c is cost per unit effort
K is ecosystem carrying capacity

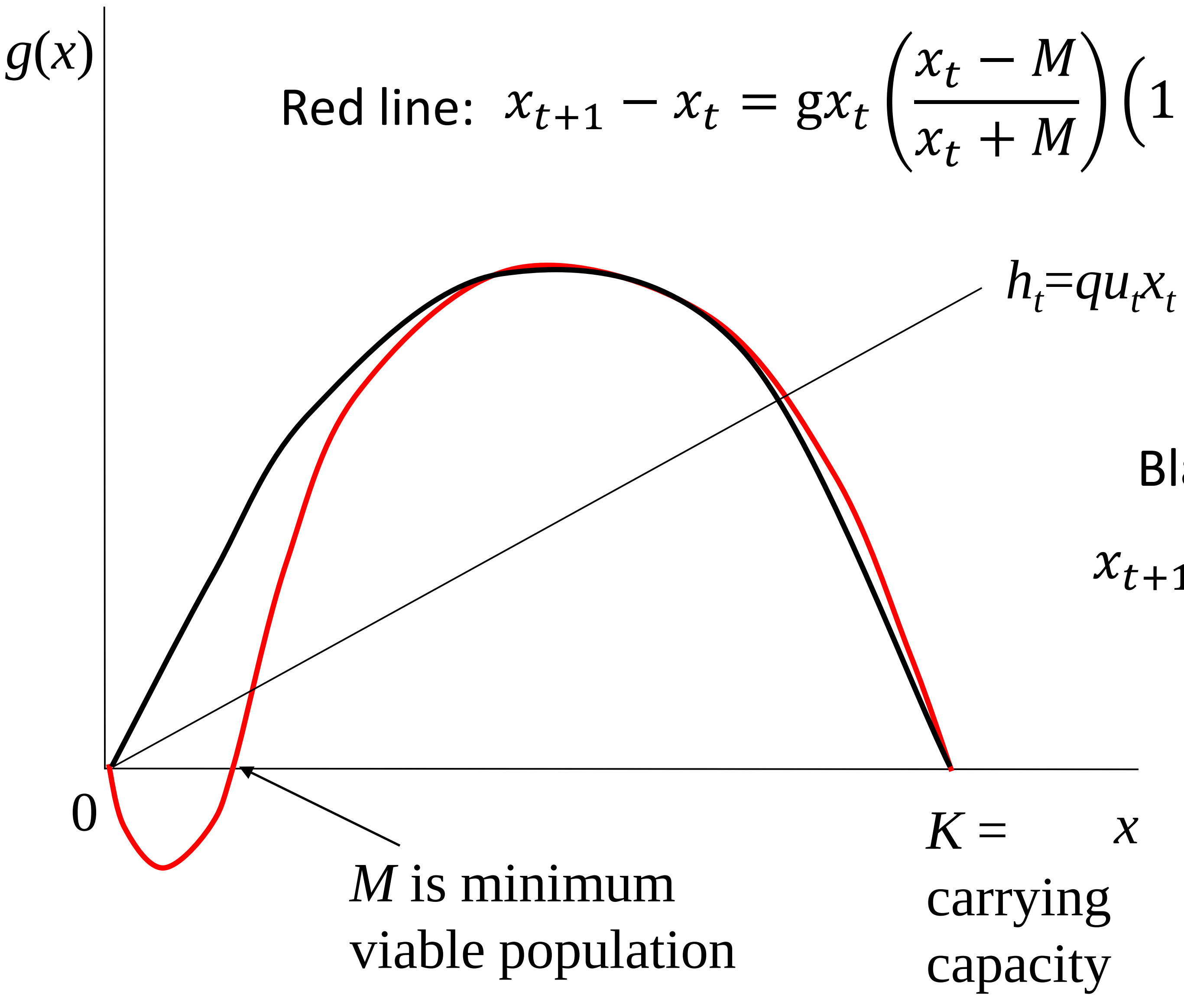


Logistic growth function: $g(x_t)$

q is a catchability coefficient
u_t is effort (control variable)
g is intrinsic growth rate
K is ecosystem carrying capacity



Growth Functions:



Red line: $x_{t+1} - x_t = gx_t\left(\frac{x_t - M}{x_t + M}\right)\left(1 - \frac{x_t}{K}\right)$

Black line:

$$x_{t+1} - x_t = gx_t\left(1 - \frac{x_t}{K}\right)$$

Blue line:

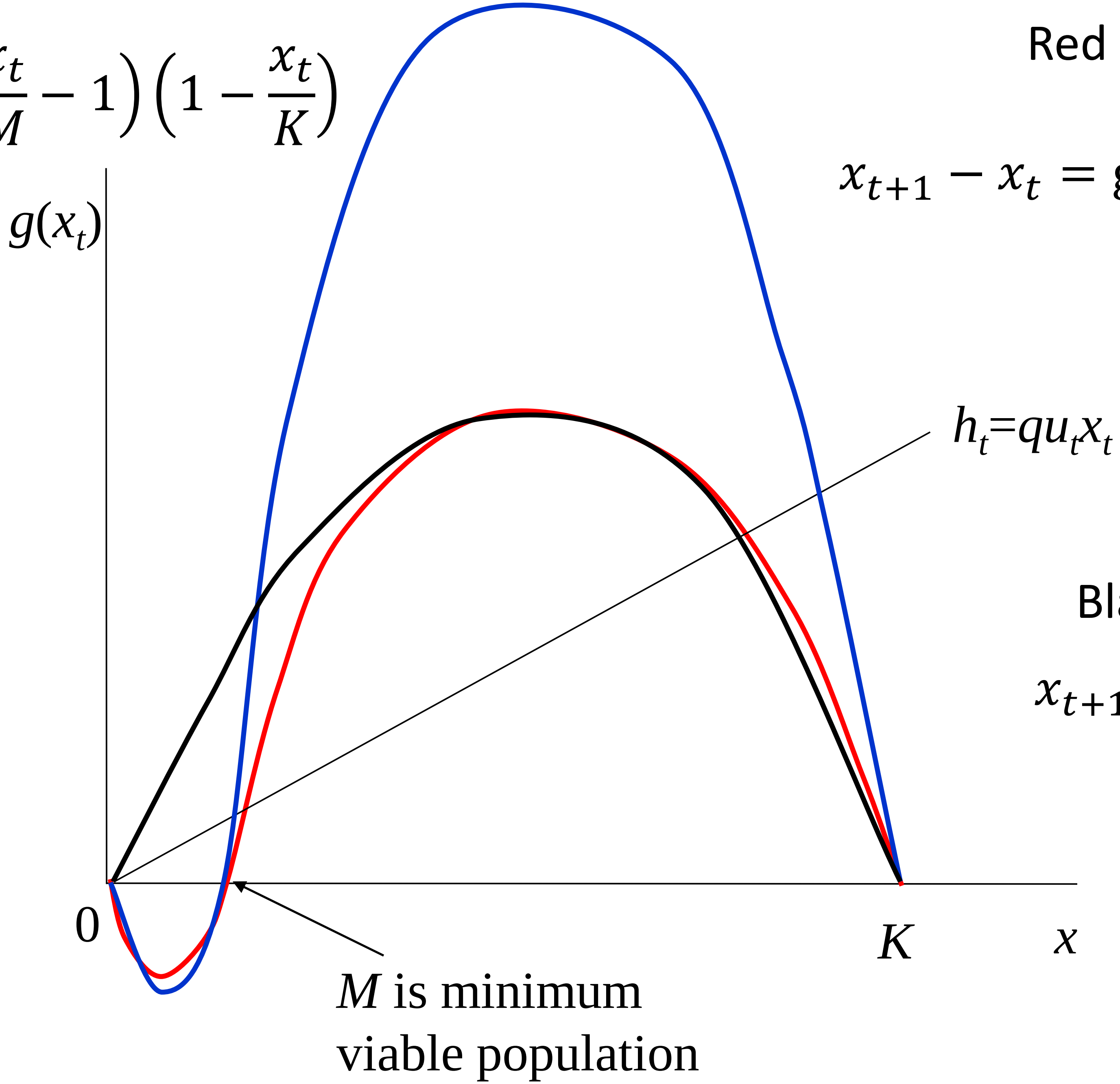
$$x_{t+1} - x_t = \mathrm{g}x_t\left(\frac{x_t}{M} - 1\right)\left(1 - \frac{x_t}{K}\right)$$

Red line:

$$x_{t+1} - x_t = \mathrm{g}x_t\left(\frac{x_t - M}{x_t + M}\right)\left(1 - \frac{x_t}{K}\right)$$

Black line:

$$x_{t+1} - x_t = \mathrm{g}x_t\left(1 - \frac{x_t}{K}\right)$$



R code for two functions compared to logistics:

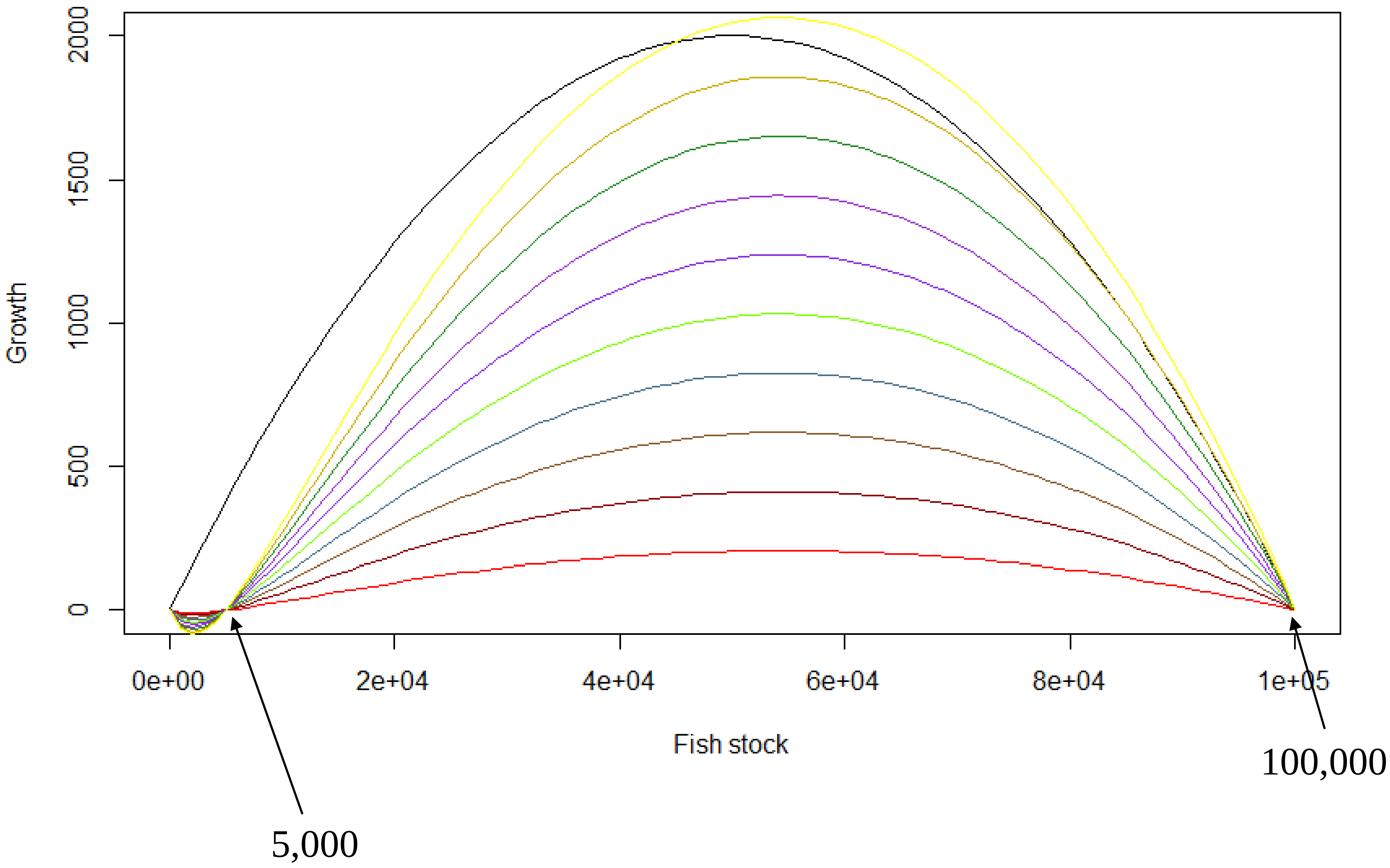
$$(1)\ x_{t+1} - x_t = r_2 x_t \left(\frac{x_t - M}{x_t + M}\right)\left(1 - \frac{x_t}{K}\right)$$

$$(2)\ x_{t+1} - x_t = r_3 x_t \left(\frac{x_t}{M} - 1\right)\left(1 - \frac{x_t}{K}\right)$$

```
M=5000; K = 100000  # minimum viable population and carrying capacity
r1 = 0.08
r2 <- seq(0.01, 0.1, 0.01)
r3 <- seq(0.001, 0.008, 0.001)
clrs <-c('red', 'red4', 'tan4', 'skyblue4', 'chartreuse', 'blueviolet','darkorchid', 'forestgreen','gold3','yellow’)

# Plot the initial logistics function
curve(r1*x*(1-x/K), 0, K, xlab = 'Fish stock', ylab='Growth', col='black’)

# Plot the function (1) on top of it
for (i in 1:10) { curve(r2[i]*x*((x-M)/(x+M))*(1-x/K), add=TRUE, col=clrs[i]) }

# After re-running the logistics function, plot the function (2) on top of it
for (i in 1:8) { curve(r3[i]*x*(x/M-1)*(1-x/K), add=TRUE, col=clrs[i]) }
```

$0.01 \leq g \leq 0.1$, where $g = 0.08$ for logistics

$$x_{t+1} - x_t = g x_t \left(\frac{x_t - M}{x_t + M}\right)\left(1 - \frac{x_t}{K}\right)$$



$0.001 \le g \le 0.008$, where $g = 0.08$ for logistics

$$x_{t+1} - x_t = g x_t \left(\frac{x_t}{M} - 1\right)\left(1 - \frac{x_t}{K}\right)$$