Chapter 18
Linear, First-Order Difference Equations

In this chapter we will learn how to solve autonomous and non-autonomous linear, first-order difference equations. The linear autonomous first-order difference equation has following general form:

\[ y_{t+1} = ay_t + b, \ t = 0, 1, 2, \ldots \quad (18.1) \]

where \( a \) and \( b \) are known constants. (18.1) can be solved using **iterative method** and **guess and verify method**. In the iterative method we start from the first period \( (t = 0) \) and solve forward.
The solution is

\[ y_t = a^t y_0 + b \left( \frac{1 - a^t}{1 - a} \right), \text{ if } a \neq 1 \]  \hspace{1em} (18.2)

and

\[ y_t = y_0 + bt \text{ if } a = 1 \]  \hspace{1em} (18.3)

(18.2) can also be written as

\[ y_t = a^t \left( y_0 - \frac{b}{1-a} \right) + \frac{b}{1-a} \]  \hspace{1em} (18.4)

where as we will see below that \( \frac{b}{1-a} \) is steady state value of \( y \). Also steady state exists only in the case \( a \neq 1 \).

If we know the initial condition \( (y_0) \) we can use iterative method. In the case, we don’t know the initial condition we can still solve the difference equation using guess and
verify method. The general approach is very much identical to the one we used in solving first order linear autonomous differential equation.

We divide the difference equation in two parts: steady state part and homogeneous part. The steady state solution $\bar{y}$ is given by

$$\bar{y} = a\bar{y} + b$$

which implies

$$\bar{y} = \frac{b}{1 - a} \quad (18.5)$$

Notice that steady-state solution exists only when $a \neq 1$.

The homogeneous part is given by

$$y_{t+1} = ay_t. \quad (18.6)$$

Now assume that the solution of homogeneous part has following form:
\[ y_h = Cd^t \]  \hspace{1cm} (18.7)

where \( C \) and \( d \) are two undetermined coefficients. Basically we have to find \( C \) and \( d \) which satisfy the differential equation (18.1). To do this, put (18.7) in (18.6), we get

\[ Cd^{t+1} = aC^td^t \]

which implies

\[ d = a. \]  \hspace{1cm} (18.8)

Thus the general solution for the homogeneous part is given by

\[ y_h = Ca^t. \]  \hspace{1cm} (18.9)

The complete solution for the differential equation (18.1) is given by

\[ y = y_h + \bar{y} \]

which implies
\[ y_t = Ca^t + \frac{b}{1-a} \text{ if } a \neq 1. \quad (18.10) \]

**Important Remark:** The form of (18.10) is slightly different from what is given in the textbook (Theorem 18.2). In the textbook, the solution is of the form

\[ y_t = C_1 a^t + b \left( \frac{1-a^t}{1-a} \right) \quad (18.11) \]

where \( C_1 \) is some constant. (18.11) can be rewritten as

\[ y_t = C_1 \left( 1 - \frac{b}{1-a} \right) a^t + \frac{b}{1-a}. \quad (18.12) \]

Thus

\[ C = C_1 \left( 1 - \frac{b}{1-a} \right). \quad (18.13) \]
If we know, the initial value $y_0$ then we can pin down $C$ which turns out to be equal to

$$C = y_0 - \frac{b}{1 - a}.$$  \hspace{1cm} (18.14)

From (18.10) it is immediately clear that the solution will converge to steady state if and only if $|a| < 1$.

In the case, $a = 1$ steady-state does not exist and one has to modify the guess. The homogeneous solution is still given by $y_h = Ca^t$. But with $a = 1$,

$$y_h = C.$$ \hspace{1cm} (18.15)

In order to find solution for non-homogeneous part assume that

$$y_{nh} = kt.$$ \hspace{1cm} (18.16)

Putting (18.16) in (18.1), we get

$$k(t + 1) = kt + b$$
which implies

\[ k = b. \]

Thus, the solution for non-homogeneous part is

\[ y_{nh} = bt. \]  \hspace{1cm} (18.17)

The complete solution in the case \( a = 1 \) is given by, \( y_t = y_h + y_{nh} \) which implies

\[ y_t = C + bt. \]  \hspace{1cm} (18.18)
Non-autonomous Equation

The general form of linear, non-autonomous, first-order difference equation is

\[ y_{t+1} = a_t y_t + b_t, \quad t = 0, 1, 2, \ldots \]  \hspace{1cm} (18.19)

By using \textbf{iterative method} one can show that the solution is given by

\[ y_t = \prod_{i=0}^{t-1} a_i y_0 + \sum_{k=0}^{t-1} b_k \prod_{i=k}^{t-1} \frac{a_i}{a_k}. \]  \hspace{1cm} (18.20)