Chapter 24
Simultaneous Systems of Differential Equations

We will learn how to solve system of first-order linear and nonlinear autonomous differential equations. Such systems arise when a model involves two and more variable. A single differential equation of second and higher order can also be converted into a system of first-order differential equation.

Linear Differential Equation Systems

A linear system of two autonomous first-order differential equations is expressed as

\[ \dot{y}_1 = a_{11} y_1 + a_{12} y_2 + b_1 \]  \hspace{1cm} (24.1)

\[ \dot{y}_2 = a_{21} y_1 + a_{22} y_2 + b_2 \]  \hspace{1cm} (24.2)
Both equations have to be solved together. There are two methods to solve them:

1. **Substitution Method:** Convert the system into a single linear differential equation and then solve it using methods developed in earlier chapters.

2. **Direct Method:** Use guess and verify method directly.

Remark: In both methods, we divide the differential equation system in two parts: homogeneous part and steady-state part. Complete solution is given by

\[ y_1 = y_1^h + \bar{y}_1 \]

\[ y_2 = y_2^h + \bar{y}_2 \]
Substitution Method

We want to solve system of differential equations given by (24.1) and (24.2). Let us solve the homogenous part first. The homogenous part of the system is given by

\[ \dot{y}_1 = a_{11} y_1 + a_{12} y_2 \]  
(24.3)

\[ \dot{y}_2 = a_{21} y_1 + a_{22} y_2 \]  
(24.4)

First step is to convert this system in single differential equation. Differentiate (24.3) with respect to time, we have

\[ \ddot{y}_1 = a_{11} \dot{y}_1 + a_{12} \dot{y}_2. \]  
(24.5)

Putting (24.4) in (24.5), we have

\[ \ddot{y}_1 = a_{11} \dot{y}_1 + a_{12} (a_{21} y_1 + a_{22} y_2). \]  
(24.6)

Further (24.3) can be rewritten as
\[ y_2 = \frac{\dot{y}_1 - a_{11}y_1}{a_{12}}. \quad (24.7) \]

Putting (24.7) in (24.6), we have

\[ \ddot{y}_1 - (a_{11} + a_{22})\dot{y}_1 + (a_{11}a_{22} - a_{12}a_{21})y_1 = 0 \quad (24.8) \]

which is a linear autonomous second-order differential equation, which can be solved using methods discussed in chapter 23. Let \( a_1 = -(a_{11} + a_{22}) \) and \( a_2 = a_{11} + a_{22} \), then (24.8) can be written as

\[ \ddot{y} + a_1\dot{y} + a_2y = 0. \quad (24.9) \]
Solutions are

\[ y_1^h(t) = c_1 \exp^{r_1 t} + c_2 \exp^{r_2 t}, \text{ if } r_1 \neq r_2 \]

and roots are real.

\[ y_1^h(t) = (c_1 + t c_2) \exp^{r t}, \text{ if } r_1 = r_2 = r \]

\[ y_1^h(t) = B_1 \exp^{ht} \cos vt + B_2 \exp^{ht} \sin vt, \]

if \( r_1 \neq r_2 \) and roots are complex.

Once, we have found \( y_1^h(t) \), we can find \( y_2^h(t) \) using (24.7), which is

\[ y_2 = \frac{\dot{y}_1 - a_{11} y_1}{a_{12}}. \]
The solutions are

Case I: \( r_1 \neq r_2 \) and roots are real:

\[
y_h^2(t) = \frac{r_1 - a_{11}}{a_{12}} c_1 \exp^{r_1 t} + \frac{r_2 - a_{11}}{a_{12}} c_2 \exp^{r_2 t}
\]

Case II: \( r_1 = r_2 = r \) (Repeated Roots):

\[
y_h^2(t) = \left[ \frac{r - a_{11}}{a_{12}} (c_1 + t c_2) + \frac{c_2}{a_{12}} \right] \exp^{r t}
\]

Case III: \( r_1 \neq r_2 \) and roots are complex:

\[
y_h^2(t) = \exp^{ht} \left[ \frac{(h - a_{11}) B_1 + v B_2}{a_{12}} \cos vt \right] + \exp^{ht} \left[ \frac{(h - a_{11}) B_2 - v B_1}{a_{12}} \sin vt \right]
\]
Steady-State Solution

To find steady-state solution, we set \( \dot{y}_1 = \dot{y}_2 = 0 \) in equations (24.1) and (24.2). The resulting equations are

\[
a_{11} \bar{y}_1 + a_{12} \bar{y}_2 + b_1 = 0 \tag{24.10}
\]

\[
a_{21} \bar{y}_1 + a_{22} \bar{y}_2 + b_2 = 0 \tag{24.11}
\]

The solutions are

\[
\bar{y}_1 = \frac{a_{12} b_2 - a_{22} b_1}{a_{11} a_{22} - a_{12} a_{21}} \tag{24.12}
\]

\[
\bar{y}_2 = \frac{a_{21} b_1 - a_{11} b_2}{a_{11} a_{22} - a_{12} a_{21}} \tag{24.13}
\]
By combining steady-state solutions and homogeneous solutions, we get complete solution. In order to derive values of arbitrary constants $c_1, c_2, B_1, B_2$, we need to specify initial conditions. Usually, we are given values of $y_1(0)$ and $y_2(0)$.

**Important Remarks**

1. A system of $n$ first-order linear differential equations can be converted into a single differential linear equation of order $n$

2. Converse is also true. A single linear differential equation of order $n$ can be converted in a system of $n$ first-order linear differential equations.
Stability of Steady State

In order to know, whether steady state is stable or not, we need to find out under what conditions

$$\lim_{t \to \infty} y_1 = \bar{y}_1 \& \lim_{t \to \infty} y_2 = \bar{y}_2 \quad (24.14).$$

We have to consider three cases:

**Case I:** $r_1 \neq r_2$ and roots are real. In this case, we require $r_1 \& r_2 < 0$. Otherwise, the steady-state is unstable.

**Case II:** $r_1 = r_2 = r$. In this case, we require that $r < 0$.

**Case III:** $r_1 \neq r_2$ and roots are complex. In this case we require that the real part of roots $h < 0$. 

\[9\]
An Important Exception to Case I

If one of the characteristic roots is positive and the other negative, the steady-state equilibrium is called a **saddle point** equilibrium. It is unstable. However, in a very particular case, $y_1$ and $y_2$ converge to their steady state solutions if the initial conditions for $y_1$ and $y_2$ satisfy following equation:

$$y_2(0) = \frac{r_1 - a_{11}}{a_{12}} (y_1(0) - \bar{y}_1) + \bar{y}_2 \quad (24.15)$$

where $r_1$ is the negative root and $r_2$ is the positive root. The locus of points $(y_1, y_2)$ defined by equation (24.15) is known as the **saddle path**.
The Direct Method

Let

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \]

\[ b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \dot{y} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} \]

Then the system of linear differential equations given in (24.1) and (24.2) can be written in matrix form as follows:

\[ \dot{y} = Ay + b. \quad (24.16) \]

Assume that inverse of \( A \), \( A^{-1} \), exists or determinant of \( A \), \( |A| \neq 0 \). Then steady state part of (24.16) is given by

\[ Ay + b = \hat{0} \quad (24.17) \]

where \( \hat{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). The steady-state solution is
\[ y = -A^{-1}b. \] (24.18)

**Homogeneous Part**

The homogeneous part is given by

\[ \dot{y} = Ay. \] (24.19)

In order to solve (24.19), we will use guess and verify method. Suppose that the homogeneous solutions are of the form

\[ y = k \exp^{rt} \] (24.20)

where \( k \) is \( n \)-dimensional vector and \( r \) is a scalar. If this guessed solution is correct then the first derivative of guess with respect to time \( t \) should satisfy (24.19). From (24.20), we have

\[ \dot{y} = rk \exp^{rt}. \] (24.21)

Putting (24.21) in (24.19), we have
\[ rk \exp^{rt} = Ak \exp^{rt}. \quad (24.22) \]

The above equation holds if

\[ [A - rI]k = \hat{0} \quad (24.23) \]

where \( I \) is the identity matrix. We want to find out values of \( r \) which satisfies (24.23). (24.23) is a system of linear homogeneous equations. It has nontrivial (nonzero) solution if and only if the determinant of \([A - rI]\) is equal to zero. Thus the solution values of \( r \) are found by solving

\[ |A - rI| = 0 \quad (24.24) \]

which is a polynomial equation of degree \( n \) in the unknown number \( r \). This equation is known as characteristic equation of matrix \( A \). The values of \( r \) which solve this equation are called characteristic roots or
**eigen values** of the matrix $A$. Since (24.24) is a polynomial equation of degree $n$, we will potentially have $n$ characteristic roots (or $n$ $r$s). For each $r$, there will be a distinct non-zero vector $k$ associated with (24.23). Such non-zero vector is called **eigenvector** of the matrix $A$ corresponding to the eigenvalue/characteristic root $r$. Corresponding to each distinct $r$, we will also get a distinct solution of homogeneous equation (24.19).

In the case $n = 2$, (24.24) becomes

$$
\begin{vmatrix}
  a_{11} - r & a_{12} \\
  a_{21} & a_{22} - r
\end{vmatrix}
$$

(24.25)

which simplifies to

$$
r^2 - (a_{11} + a_{22})r + (a_{11}a_{22} - a_{12}a_{21}) = 0.
$$

(24.26)

which is the same characteristic equation we obtained earlier using substitution method. (24.26) can be written as
\[ r^2 - tr(A)r + |A| = 0 \quad (24.27) \]

where \( tr(A) = a_{11} + a_{22} \) (sum of diagonal elements of \( A \)) and \( |A| = a_{11}a_{22} - a_{12}a_{21} \). The characteristic roots in this case is given by

\[
\begin{align*}
    r_1, r_2 &= \frac{tr(A)}{2} \pm \frac{1}{2} \sqrt{tr(A)^2 - 4|A|}. \quad (24.28)
\end{align*}
\]

In the case of two independent roots \( r_1, r_2 \), we will get two distinct non-zero eigenvectors \( k^1 \& k^2 \), and two distinct solutions to the system of two homogeneous differential equations \( y^1 \) and \( y^2 \). The general solution of the system then is

\[
y_h = c_1 y^1 + c_2 y^2 \quad (24.29)
\]

where \( c_1 \) and \( c_2 \) are two arbitrary constants to be determined by initial conditions.
In general, there are $n$ equations and $n$ characteristic roots; therefore there are $n$ solutions to the system of $n$ homogeneous differential equations. In the case, all characteristic roots are distinct (no repeated roots), the general solution is given by

$$y_h = c_1 y^1 + c_2 y^2 + \ldots + c_n y^n$$  \hspace{1cm} (24.30)

where $c$s are arbitrary constants.

The complete solution is given by

$$y = \bar{y} + y_h.$$  \hspace{1cm} (24.31)

**Phase Diagram**

Qualitative properties of two equation differential equation system can be analyzed using **phase diagram**, which involves plotting the two differential equations on $(y_1, y_2)$ space.
Stability Analysis Using Coefficient Matrix $A$

Let us classify the types of steady states in the system of two linear differential equations. In the case of real roots, steady-state is known as node. In the case of complex roots, steady state is known as focus. As discussed earlier, roots are given by

$$r_1, r_2 = \frac{tr(A)}{2} \pm \frac{1}{2} \sqrt{tr(A)^2 - 4|A|}. \quad (24.32)$$

Notice that repeated roots and complex roots can arise only in the case that $|A| > 0$. (24.32) implies that

$$r_1 + r_2 = a_{11} + a_{22} = tr(A) \quad (24.33)$$

and

$$r_1 r_2 = a_{11} a_{22} - a_{12} a_{21} = |A|. \quad (24.34)$$
Using above relations, we can distinguish among different types of steady states.

**Case I:** $r_1 \neq r_2$ and Real Roots

1. If $|A| < 0$, then $r_1, r_2$ have opposite signs, and we have saddle point.

2. If $|A| > 0$, then $r_1, r_2$ have same signs (positive or negative). If $r_1, r_2 < 0$, then we have stable node. If $r_1, r_2 > 0$, then we have unstable node.

**Case II:** $r_1 = r_2 = r(|A| > 0)$

1. If $r = \frac{tr(A)}{2} < 0$, we have improper stable node.

2. If $r = \frac{tr(A)}{2} > 0$, we have improper unstable node.
Case III: $r_1 \neq r_2$ and Complex Roots ($|A| > 0$)

1. If $a_{11} + a_{22} = tr(A)$ or $h < 0$, then we have **stable focus**.

2. If $a_{11} + a_{22} = tr(A)$ or $h > 0$, then we have **unstable focus**.

3. If $a_{11} + a_{22} = tr(A)$ or $h = 0$, then we have **center/vortex**.
Nonlinear Differential Equation Systems

A nonlinear system of two autonomous differential equations is expressed in general as

\[
\dot{y}_1 = F(y_1, y_2) \quad (24.35)
\]

\[
\dot{y}_2 = G(y_1, y_2). \quad (24.36)
\]

In general, it is very difficult to analytically solve non-linear systems. One can analyze its qualitative behavior using phase diagram. In addition, one can analyze its stability properties, using the techniques discussed earlier for linear systems. Steps are as follows:

Step 1. Find out steady state values of \( y_1 \) and \( y_2 \) by setting \( \dot{y}_1 = \dot{y}_2 = 0 \).

Step 2. Linearize (24.35) and (24.36) around steady state points \( \overline{y}_1, \overline{y}_2 \) using Taylor series expansion formula. In other wo-
rds, we take a first-order linear approximation to these differential equations. This implies

\[ \dot{y}_1 = F(y_1, y_2) + \frac{dF(y_1, y_2)}{dy_1}(y_1 - \bar{y}_1) + \frac{dF(y_1, y_2)}{dy_2}(y_2 - \bar{y}_2) \]  

(24.37)

and

\[ \dot{y}_2 = G(y_1, y_2) + \frac{dG(y_1, y_2)}{dy_1}(y_1 - \bar{y}_1) + \frac{dG(y_1, y_2)}{dy_2}(y_2 - \bar{y}_2). \]  

(24.38)

Since \( F(y_1, y_2) = G(y_1, y_2) = 0 \), (24.37) and (24.38) reduce to

\[ \dot{y}_1 = \frac{dF(y_1, y_2)}{dy_1}(y_1 - \bar{y}_1) + \frac{dF(y_1, y_2)}{dy_2}(y_2 - \bar{y}_2) \]  

(24.39)
and

\[ \dot{y}_2 = \frac{dG(\bar{y}_1, \bar{y}_2)}{dy_1} (y_1 - \bar{y}_1) + \frac{dG(\bar{y}_1, \bar{y}_2)}{dy_2} (y_2 - \bar{y}_2). \]  

(24.40)

Since the partial derivatives are all evaluated at a specific point \((\bar{y}_1, \bar{y}_2)\), they are all constants. Thus, (24.39) and (24.40) can be reexpressed as a system of linear differential equations.

\[ \dot{y}_1 = a_{11}y_1 + a_{12}y_2 + b_1 \]  

(24.41)

\[ \dot{y}_2 = a_{21}y_1 + a_{22}y_2 + b_2 \]  

(24.42)
Step 3: The associated coefficient matrix $A$ for (24.39) and (24.40) is

$$A = \begin{bmatrix} \frac{dF(y_1, y_2)}{dy_1} & \frac{dF(y_1, y_2)}{dy_2} \\ \frac{dG(y_1, y_2)}{dy_1} & \frac{dG(y_1, y_2)}{dy_2} \end{bmatrix}. \quad (24.43)$$

Properties of $A$ give us the behavior of the nonlinear system in the neighborhood of steady state points $(y_1, y_2)$. Since, we derive coefficient matrix $A$ by linear approximation around steady state points $(\bar{y}_1, \bar{y}_2)$, the properties of $A$ tell us only about **local stability** of nonlinear system.
24.3 Systems of Linear Difference Equations

The general form for a system of two autonomous linear difference equations is

\[ y_{t+1} = a_{11}y_t + a_{12}x_t + b_1 \]  \hspace{1cm} (24.44)

\[ x_{t+1} = a_{21}y_t + a_{22}x_t + b_2. \]  \hspace{1cm} (24.45)

Again we divide the system in two parts: homogeneous part and steady-state part and solve them separately. The sum of solutions of these parts gives us complete solution.

We can solve these systems using two methods: (i) substitution method and (ii) direct method. In substitution method, we convert the system in a single higher order difference equation. In this case, it will be second order difference equation, which we can solve using the methods discussed in Chapter 20. Under the direct method, we use guess and verify method directly.
Substitution Method

Homogeneous Solutions

The homogeneous part of the system is given by

\[ y_{t+1} = a_{11}y_t + a_{12}x_t \quad (24.46) \]

\[ x_{t+1} = a_{21}y_t + a_{22}x_t. \quad (24.47) \]

Push (24.46) one period forward and then putting (24.47) in (24.46), we get

\[ y_{t+2} = a_{11}y_{t+1} + a_{12}(a_{21}y_t + a_{22}x_t). \quad (24.48) \]

Further, (24.46) implies that

\[ x_t = \frac{y_{t+1} - a_{11}y_t}{a_{12}}. \quad (24.49) \]
Under the assumption that \( a_{12} \neq 0 \), (24.48) and (24.49) imply that

\[
y_{t+2} - (a_{11} + a_{22})y_{t+1} + (a_{11}a_{22} - a_{12}a_{21})y_t = 0
\]

(24.50)

which is a second order, linear, autonomous, difference equation. The associated characteristic function is

\[
r^2 - (a_{11} + a_{22})r + (a_{11}a_{22} - a_{12}a_{21}) = 0.
\]

(24.51)

In the case of real and distinct roots,

\[
y_h = C_1 r_1^t + C_2 r_2^t.
\]

(24.52)

By putting (24.52) in (24.49), we can solve for \( x_t \), which is given by

\[
x_t = \frac{r_1 - a_{11}}{a_{12}} C_1 r_1^t + \frac{r_2 - a_{11}}{a_{12}} C_2 r_2^t.
\]

(24.53)
Steady State Solution

Steady state is given by

\[ \overline{y} = a_{11} \overline{y} + a_{12} \overline{x} + b_1 \]  \hspace{1cm} (24.54)

\[ \overline{x} = a_{21} \overline{y} + a_{22} \overline{x} + b_2 \]  \hspace{1cm} (24.55)

The solution are

\[ \overline{y} = \frac{(1 - a_{11})b_1 + a_{12}b_2}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} \]  \hspace{1cm} (24.56)

\[ \overline{x} = \frac{a_{21}b_1 + (1 - a_{11})b_2}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} \]  \hspace{1cm} (24.57)

Here we assume that the denominator is not equal to zero.
Direct Method

Let

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \]

\[ b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad Y = \begin{bmatrix} y \\ x \end{bmatrix}, \quad k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \quad \hat{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Homogeneous Solution

The homogeneous system given in (24.46) and (24.47) can be written as

\[ Y_{t+1} = AY_t. \quad (24.58) \]

Assume that the solution of (24.58) are of following form

\[ Y_t = kr^t \quad (24.59) \]

where \( k \) is a vector of arbitrary constants and \( r \) is a scalar. Putting (24.59) in (24.58), we have
\[ kr^{t+1} = Ak^t. \quad (24.60) \]

In order to derive non-trivial \( r \neq 0 \) we need to solve a system of linear equation given by

\[ (A - rI)k = \hat{0} \quad (24.61) \]

where \( I \) is an identity matrix. The values of \( r \) is found by solving

\[ |A - rI| = \hat{0}. \quad (24.62) \]

The values of \( r \) which solve (24.62) are known as characteristic roots.

In general, if the system consists of \( n \) equations, there will be \( n \) characteristic roots. For each characteristic root, there will be an associated vector \( k \) (eigen vector). The general solution to the homogeneous form is a linear combination of \( n \) distinct roots.
Stability of Steady State

Theorem 24.9: A system of two autonomous linear difference equations is asymptotically stable if and only if the absolute values of both characteristic roots are less than unity.