Lecture 2

General equilibrium Models: Finite Period Economies

1. Introduction

In macroeconomics, we study behavior of economy-wide aggregates – e.g. GDP, savings, investment, employment and so on - and their interrelations. The behavior of aggregates and their interrelations are results of decisions and interactions of consumers and firms in different markets – goods market, labor market, and asset market. In addition, most of the issues in macroeconomics are inherently dynamic. In growth we are concerned with the behavior of output, investment, and consumption over long run. Business cycle relates to short-run movements in a number of variables *e.g.* GDP, employment, real wage, inflation, investment. Savings involve foregoing current consumption for the sake of higher future consumption. Investment decision requires comparison of current costs with expected future returns. Thus in macroeconomics, we are concerned with the behavior of agents across time and markets *i.e.* macroeconomics is about dynamics and general equilibrium.

In order to analyze macroeconomics issues, we need a framework which can handle both general equilibrium and dynamics. **Dynamic general equilibrium (DGE)** models provide one such framework. These models ensure that aggregate or economy-wide variables are consistent with the decisions and interactions of individual agents, and the decisions of individual agents are optimal given aggregate variables and other parameters.

In this lecture, we will develop a basic framework of these models and study some of its applications. DGE models generally have following building blocks:

1. **Description of the Economy/ Environment:** This section gives details about number and types of goods and agents, preferences (objective functions) of agents, their endowments, technology, structure of markets, trading processes, information structure, timing of events, time period, and sources of shocks. It is extremely important to clearly describe the economy. Basically this section lays out the structure of the economy and all the assumptions a modeler makes.

2. **Optimal Decisions of Agents:** This section analyzes optimal behavior of agents subject to given constraints *e.g.* consumers maximize utility subject to their budget constraints, firms maximize profit. This is partial equilibrium analysis. At this stage, it is very important to differentiate between what variables are choice variables of the agents and what variables they take as given.

   There are two types of variables - endogenous variables and exogenous variables. Endogenous variables are variables whose solution we are seeking. Exogenous variables are variables given from outside. Endogenous variables are also of two types : (i) individual choice variables, which individual agents choose; and (ii) aggregate or economy-wide endogenous variables, which are not chosen by individual agents, but are the outcomes of their interactions. For example, in the competitive market consumption by a consumer or production by a firm is individual choice variable, but the prices are aggregate endogenous variables. Individual agents while making decisions take exogenous and aggregate endogenous variables as given.

3. **Definition of Equilibrium:** This section tells us what constitutes an equilibrium. Usually equilibrium consists of a system of prices and allocations which are consistent with the optimizing behavior of agents given the market structure and feasibility constraints (coming from endowments, technology etc.).

4. **Solution:** Ultimately, we are interested in the solutions of the model, which involves expressing endogenous variables solely as functions of exogenous variables. The optimal decisions of agents together with the definition of equilibrium allow us to find solutions of the model.

To illustrate these elements, we will solve many examples. But first we are going to analyze some partial equilibrium models. We will assume that agents while making decisions take market prices as given.

1. Partial Equilibrium Models

Example 1

Suppose that the utility function of a consumer is $U(c_1, c_2)$, where c_1 and c_2 are consumption of good 1 and good 2 respectively. The utility function is an increasing and concave function of consumption of both goods. The budget constraint faced by the consumer is $p_1 c_1 + p_2 c_2 = Y$, where p_1, p_2, and Y are prices of good 1 and good 2 and income respectively. We want to find out the optimal choices of c_1 and c_2 (consumption bundle) given prices and income.

The consumer's problem is

$$\max_{c_1, c_2} U(c_1, c_2)$$

subject to

$$p_1 c_1 + p_2 c_2 = Y. \tag{1.1}$$

The easiest way to solve this problem is to put the budget constraint in the utility function (or the objective function). This way we convert the constrained optimization problem in an unconstrained optimization problem. Then, we have

$$\max_{c_1} U(c_1, \frac{Y}{p_2} - \frac{p_1}{p_2} c_1). \tag{1.2}$$

The first order condition is

$$U_1 = \frac{p_1}{p_2} U_2 \tag{1.3}$$

which can be rewritten as

$$\frac{U_1}{U_2} = \frac{p_1}{p_2}. \tag{1.4}$$

(1.4) equates the marginal rate substitution between the two goods to the ratio of their prices. Using equations (1.1) and (1.4) we can derive consumption functions $c_1(p_1, p_2, Y)$ and $c_2(p_1, p_2, Y)$.

Let us take an specific example. Suppose that $U(c_1, c_2) = \ln c_1 + \ln c_2$. Then from (1.4) we have

$$p_1c_1 = p_2c_2. \tag{1.5}$$

Putting (1.5) in the budget constraint we have $c_1 = \frac{1}{2}\frac{Y}{p_1}$. Then (1.5) implies that $c_2 = \frac{1}{2}\frac{Y}{p_2}$.

Example 2

Let us take another example with the labor-leisure choice. Suppose that the utility function of the consumer is $U(c, 1-l)$ where l is the amount of time worked ($1-l$ is the leisure). Assume that the utility function is an increasing and concave function of consumption and leisure. Let P and W be the price of the consumption good and wage respectively. What will be the optimal choices of consumption and leisure (labor supply) given prices?

The consumer's problem is

$$\max_{c,l} U(c, 1-l)$$

subject to

$$Pc = Wl. \tag{1.6}$$

Putting (1.6) in the objective function we have

$$\max_{l} U(Wl/P, 1-l). \tag{1.7}$$

The first order condition is

$$\frac{U_2}{U_1} = \frac{W}{P} \tag{1.8}$$

which equates the MRS between consumption and leisure to the real wage. Using (1.6) and (1.8) we can derive the individual demand function $c(W, P)$ and the labor supply function $l(W, p)$.

Example 3

Let us now take a two-period model where consumers face consumption-savings choi- ces. Suppose that the utility function of a consumer is $U(c_1, c_2)$, where c_1 and c_2 are consumption in time period 1 and 2 respectively. The consumer can save in terms of financial instrument at the net rate of interest

r. Let Y be the income in the first period. What will be the optimal choices of consumption and savings given the rate of interest?

The consumer's problem is

$$\max_{c_1, c_2, s} U(c_1, c_2)$$

subject to

$$c_1 + s = Y \ \& \tag{1.9}$$

$$c_2 = (1 + r)s \tag{1.10}$$

where s is the amount of saving in period 1.

By putting (1.9) and (1.10) in the objective function we have

$$\max_{s} U(Y - s, (1 + r)s).$$

The first order condition is

$$\frac{U_1}{U_2} = 1 + r \tag{1.11}$$

which equates the MRS between consumption in two periods to the gross rate of interest.

Using (1.11) and the budget constraints, we can derive consumption functions $c_1(r, Y)$ and $c_2(r, Y)$ and savings function $s(r, Y)$.

Example 4

Let us modify the previous problem as follows. Suppose that the consumer also has access to production technology which converts k units investment in period one to $f(k)$ units of goods in the second period. The production technology is an increasing and concave function of k. Now the consumer faces a **portfolio-choice problem**. It can enhance second period consumption by savings in the financial instrument or it can invest. What will be the optimal portfolio?

The consumer's problem is

$$\max_{c_1, c_2, s, k} U(c_1, c_2)$$

subject to

$$c_1 + s + k = Y \ \& \tag{1.12}$$

$$c_2 = (1+r)s + f(k). \tag{1.13}$$

By putting (1.12) and (1.13) in the objective function we have

$$\max_{s,k} U(Y - s - k, (1+r)s + f(k)). \tag{1.14}$$

The first order conditions are

$$s : \frac{U_1}{U_2} = 1 + r \ \& \tag{1.15}$$

$$k : \frac{U_1}{U_2} = f_k(k). \tag{1.16}$$

Combining (1.15) and (1.16), we have

$$f_k(k) = 1 + r \tag{1.17}$$

which equates the marginal product of capital to the gross rate of interest. Since the consumer now has two instruments of savings, at the optimum it must be indifferent between the two. Using (1.12)-(1.16) one can derive the consumption functions $c_1(r, Y)$ and $c_2(r, Y)$ and savings $s(r, Y)$ and investment functions $k(r, Y)$.

Example 5

Suppose that there is a firm. The production depends on investment, k, and labor input, l. More specifically, the production function, $f(k, l)$ is an increasing and concave function of investment and labor input. Let w, r, and δ be the real wage, the net rate of interest, and the rate of depreciation respectively. What would be the optimal choices of k and l?

The objective of the representative firms is to choose k and l in order to maximize the profit

$$PR \equiv f(k, l) + (1 - \delta)k - (1 + r)k - wl = f(k, l) - (\delta + r)k - wl. \tag{1.18}$$

The first order conditions are

$$MPK \equiv f_k(k,l) = r + \delta \ \& \tag{1.19}$$

$$MPL \equiv f_l(k,l) = w. \tag{1.20}$$

Using (1.19) and (1.20) we can derive the input demand functions $k(w,r)$ and $l(w,r)$. So far we have analyzed several partial equilibrium models. Let us now turn to general equilibrium models.

2. General Equilibrium Models

Let us begin with a two-period representative agent economy with production. By assumption, consumers in the economy have identical preferences and identical wealth levels.

Example 6

Environment

1. Economy lasts for two periods.

2. Single commodity.

3. Large number (unit measure) of identical consumers and identical firms.

4. Consumers own all firms equally.

5. In the first period each consumer is endowed with y units of good, which they can consume and save. They have zero endowment in the second period.

6. The preference of the 'representative consumer' is given by

$$U = \ln c_1 + \beta \ln c_2 \tag{2.1}$$

   where c_i is the consumption at time i and $\beta \in (0,1)$ is the discount rate.

7. The ‘representative firm’ possesses a technology which converts k units investment in period one to k^α units of goods in the second period. Let δ be the rate of depreciation.

8. Tradings between consumers and firms take place in a competitive market.

Consumer Optimization

Let us first state the budget constraint of the representative consumer. The first period budget constraint is given by

$$c_1 + s = y \tag{2.2}$$

where s is the saving. The second period constraint is given by

$$c_2 = s(1+r) + PR \tag{2.3}$$

where r is the real rate of interest (taken as given by consumers) and PR is the profit repatriated by the representative firm to the representative consumer. We will define PR below. We can combine (2.2) and (2.3) and get inter-temporal budget constraint of the consumer given by

$$c_1 + \frac{c_2}{1+r} = y + \frac{PR}{1+r}. \tag{2.4}$$

The consumer problem is

$$\max_{c_1, c_2} U = \ln c_1 + \beta \ln c_2 \tag{2.5}$$

subject to the inter-temporal budget constraint in (2.4). Let λ be the Langrangian multiplier associated with (2.4), then the first order conditions are

$$c_1: \;\; MU_1 \equiv \frac{1}{c_1} = \lambda \tag{2.6}$$

$$c_2: \;\; MU_2 \equiv \frac{\beta}{c_2} = \frac{\lambda}{1+r}. \tag{2.7}$$

Combining (2.6) and (2.7), we have

$$\frac{MU_1}{MU_2} \equiv MRS \equiv -\frac{dc_2}{dc_1} \equiv \frac{c_2}{\beta c_1} = 1 + r \tag{2.8}$$

where MRS is the marginal rate of substitution. (2.8) together with the budget constraint gives the consumption functions:

$$c_1 = \frac{1}{1+\beta}\left[y + \frac{PR}{1+r}\right] \tag{2.9}$$

$$c_2 = \frac{\beta(1+r)}{1+\beta}\left[y + \frac{PR}{1+r}\right]. \tag{2.10}$$

(2.9) and (2.10) show that the current consumption is a function of the life-time income and not only the current income. This illustrates the **permanent income hypothesis**. Savings/borrowings allow a consumer to consume more or less than the current income in a given period.

Firm Optimization

The objective of the representative firms is to choose k in order to maximize the profit

$$PR \equiv k^{\alpha} + (1-\delta)k - (1+r)k = k^{\alpha} - (\delta + r)k. \tag{2.11}$$

To simplify the problem, we will assume that $\delta = 1$ (100% depreciation). The first order condition yields

$$MPK \equiv \alpha k^{\alpha-1} = 1 + r \tag{2.12}$$

Definition of The Equilibrium

Competitive Equilibrium: A competitive equilibrium is the price (real rate of interest) r and allocation $\{c_1,\ c_2,\ k\}$ such that:

(a) the representative consumer maximizes its utility given prices and subject to its budget constraints;

(b) the representative firm maximizes profit given prices and technology; and

(c) supply equals demand for each good:

$$c_1 + k = y,\ c_2 = k^{\alpha}. \tag{2.13}$$

The last part of the definition pins down the equilibrium level of real rate of interest, r. In order to get equilibrium allocation and prices, use (2.8), (2.12), and market clearing condition (2.13). After some work, you can show that the equilibrium allocation and the real rate of interest satisfy:

$$k = \frac{\alpha\beta y}{1+\alpha\beta} \tag{2.14}$$

$$c_1 = \frac{1}{1+\alpha\beta} y \tag{2.15}$$

$$c_2 = \left[\frac{\alpha\beta y}{1+\alpha\beta}\right]^{\alpha} \tag{2.16}$$

$$r = \alpha\left[\frac{1+\alpha\beta}{\alpha\beta y}\right]^{1-\alpha} - 1. \tag{2.17}$$

Example 7
Heterogeneity: A Model of Private Debt/Credit

In the previous example, we assumed that all agents are alike. Let us now introduce heterogeneity. Agents can be heterogeneous in terms of their preferences, endowments, information etc. We will consider a simple model in which agents are heterogenous in terms of their endowment pattern. This will allow us to examine the issue of the circulation credit and debt.

Suppose that there are two types of individuals: borrowers, with no endowment in the first period and endowment y in the second period, and lenders with endowment y in the first period and no endowment in the second period. With this structure of endowment, borrowers would like to borrow in the first period while lenders would like to lend in order to finance their consumption in the second period.

Preferences and Constraints of Lenders

Let $c_{1,l}, c_{2,l}$, and l denote the first-period consumption, second-period consumption, and the amount of lending of a lender respectively. The first-period budget constraint for a lender is

$$c_{1,l} + l = y. \tag{2.18}$$

The second-period budget constraint is

$$c_{2,l} = (1 + r)l. \tag{2.19}$$

The life-time budget constraint is given by

$$c_{1,l} + \frac{c_{2,l}}{1 + r} = y. \tag{2.20}$$

The lender chooses $c_{1,l}, c_{2,l}, l$ in order to maximize

$$U(c_{1,l}, c_{2,l}) \tag{2.21}$$

subject to its budget constraints.

Optimal Choices of Lender

Putting (2.18) and (2.19) in (2.21), we have

$$\max_{l} U(y - l, (1 + r)l). \tag{2.22}$$

The first order condition is

$$\frac{U_1(c_{1,l}, c_{2,l})}{U_2(c_{1,l}, c_{2,l})} = 1 + r. \tag{2.23}$$

(2.23) equates the marginal rate of substitution between the current and the future consumption to the rate of interest. Using this equation, we can derive the amount lent, l, as a function of interest rate r, $l(r)$. Normally we assume that utility function is such that lending, l, is an increasing function of the real interest rate, r, *i.e.*, $l_1(r) > 0$. Using (2.18), (2.19), and (2.23), we can derive $c_{1,l}, c_{2,l}, l$ as a function of interest rate r.

Preferences and Constraints of Borrowers

Let $c_{1,b}, c_{2,b}$, and b denote the first-period consumption, second-period consumption, and the amount of borrowing of a borrower. Let r be the net rate of interest. The first-period budget constraint for a borrower is

$$c_{1,b} = b. \tag{2.24}$$

The second-period budget constraint is

$$c_{2,b} = y - (1+r)b. \tag{2.25}$$

The life-time budget constraint is given by

$$c_{1,b} + \frac{c_{2,b}}{r} = \frac{y}{1+r}. \tag{2.26}$$

The borrower chooses $c_{1,b}, c_{2,b}, b$ in order to maximize

$$U(c_{1,b}, c_{2,b}) \tag{2.27}$$

subject to its budget constraints.

Optimal Choices of Borrower

Putting (2.24) and (2.25) in (2.27), we have

$$\max_{l} U(b, y - (1+r)b). \tag{2.28}$$

The first order condition is

$$\frac{U_1(c_{1,b}, c_{2,b})}{U_2(c_{1,b}, c_{2,b})} = 1 + r. \tag{2.29}$$

(2.29) equates the marginal rate of substitution between the current and the future consumption to the real rate of interest. Using this equation, we can derive the amount borrowed, b, as a function of the real interest rate r, $b(r)$. Normally we assume that utility function is such that the borrowing, b, is a decreasing function of the interest rate, r, *i.e.*, $b_1(r) < 0$. Using (2.24), (2.25), and (2.29), we can derive $c_{1,b}, c_{2,b}, b$ as a function of interest rate r.

Note that (2.23) and (2.29) equates the MRS of lenders and borrowers:

$$\frac{U_1(c_{1,l}, c_{2,l})}{U_2(c_{1,l}, c_{2,l})} = \frac{U_1(c_{1,b}, c_{2,b})}{U_2(c_{1,b}, c_{2,b})} = 1 + r. \tag{2.30}$$

Definition of The Equilibrium

Competitive Equilibrium: A competitive equilibrium is the price (real rate of interest) r and allocation $\{c_{1,l},\ c_{2,l},\ c_{1,b},\ c_{2,b},\ b,\ \&\ l\}$ such that:

(a) the representative lender maximizes its utility given prices and subject to its budget constraints;

(b) the representative borrower maximizes its utility given prices and subject to its budget constraints; and

(c) markets clear:

$$c_{1,l} + c_{1,b} = y, \;\; c_{2,l} + c_{2,b} = y, \;\; \& \;\; l(r) = b(r). \tag{2.31}$$

The condition that $l(r) = b(r)$ allows us to pin down the equilibrium rate of interest. Once we have determined the equilibrium real rate of interest, we can derive the allocations $c_{1,l}, c_{2,l}, l, c_{1,b}, c_{2,b}, b, l$.

3. The Social Planner Problem

For policy formulation, it is important to know whether the allocation made by the market is efficient and maximize social welfare. If market allocation is not efficient or social welfare maximizing, then what are the options available to policy makers/ government/ social planner?

In order to know whether a particular allocation is social welfare maximizing we need to have some kind of social preference which reflects preferences of individual agents. In general, ways to aggregate preferences of individual agents are subject to debate because of differing utilities. But in the case of representative agent economy, deriving social welfare maximizing allocation is particularly simple because every agent is identical. The socially optimal allocation maximizes the representative consumer's utility subject to the resource constraint.

$$\max_{c_1, c_2, k} U = U(c_1) + \beta U(c_2)$$

subject to resource constraints

$$c_1 + k = y \; \& \tag{3.1}$$

$$c_2 = f(k) + (1 - \delta)k. \tag{3.2}$$

The first order condition is

$$k: \;\; U_1(c_1) = \beta U_2(c_2)(f_k(k) + 1 - \delta). \tag{3.3}$$

Using (3.1)-(3.3) one can derive socially optimal allocation. This allocation satisfies the condition that

$$MRS = MRPT.$$

where MRPT is the marginal rate of product transformation given by $f'(k) + 1 - \delta$.

Example 8

Let us derive the socially optimal allocation in the economy considered in example 6.

$$\max_{c_1, c_2, k} U = \ln c_1 + \beta \ln c_2$$

subject to resource constraints (assuming $\delta = 1$)

$$c_1 + k = y \tag{3.4}$$

$$c_2 = k^{\alpha} \tag{3.5}$$

One can easily show that efficient allocation is given by:

$$k = \frac{\alpha\beta y}{1 + \alpha\beta} \tag{3.6}$$

$$c_1 = \frac{1}{1 + \alpha\beta} y \tag{3.7}$$

$$c_2 = \left[\frac{\alpha\beta y}{1 + \alpha\beta}\right]^{\alpha} \tag{3.8}$$

which coincides with market allocation (2.14, 2.15, and 2.16).

Why do social optimal allocations coincide with market allocations? From microeconomics, we know that socially optimal allocations are also **Pareto optimal or efficient**. An allocation (in our case $\{c_1\ c_2\ k\}$) is *Pareto optimal* or Pareto efficient if production and distribution cannot be reorganized to increase the utility of one or more individuals without decreasing utility of others.

From the **first and second fundamental theorems of welfare economics** we know that competitive allocations are Pareto optimal (under certain conditions) and optimal allocations can be supported as competitive equilibria (under more restrictive conditions). Our example (actually all examples considered so far) satisfies conditions under which fundamental theorems apply and thus market allocation coincides with social optimal allocation.

In the competitive economies where the second fundamental theorem of welfare applies, usually it is easier to compute competitive equilibrium by solving the social planner problem, rather than going through the consumers and firms optimization problem and imposing the market clearing conditions. Steps involved in computing competitive equilibrium through this method are as follows:

1. Compute the socially optimal allocation.

2. Derive the real rate of interest by equating it to either MRS or MRPT and evaluating derivatives at the optimal allocation:

$$MRS \equiv \frac{MU_1}{MU_2} = MRPT \equiv MPK = 1 + r. \tag{3.9}$$

3. Other prices such as wages can be computed by evaluating the relevant MRS at the socially optimal allocation.

We can use the above method to compute the competitive equilibrium in economies which satisfy conditions of second fundamental theorem of welfare. However, there are many economies which do not satisfy these conditions. In such economies, the social planner allocations normally diverge from the market allocations. Such divergence raises interesting policy issues, e.g. whether policy interventions can improve market allocations. The examples of such economies are economies with distortionary taxes, imperfect competition (e.g. Keynesian models), increasing returns, externalities, OLG economies etc.

Example 9

Let us derive the socially optimal allocation in the economy considered in example 7. Since, there is heterogeneity we have think about how to aggregate the preferences of lenders and borrowers i.e. aggregate the individual preferences into one social preference. One reasonable way is to assume that the social preference is represented by the weighted average of the individual preferences (**utilitarian social welfare function**). Let us suppose that the social planner puts weight $\lambda \in (0,1)$ on the utility of the lender and $1-\lambda$ on the utility of the borrowers. Thus, the social planner maximizes

$$\max_{c_{1,l},c_{2,l},c_{1,b},c_{2,b}} \lambda U(c_{1,l},c_{2,l}) + (1-\lambda)U(c_{1,b},c_{2,b})$$

subject to resource constraints:

$$c_{1,l} + c_{1,b} = y \ \& \tag{3.10}$$

$$c_{2,l} + c_{2,b} = y. \tag{3.11}$$

Using first order conditions, you can show that

$$\frac{U_1(c_{1,l},c_{2,l})}{U_2(c_{1,l},c_{2,l})} = \frac{U_1(c_{1,b},c_{2,b})}{U_2(c_{1,b},c_{2,b})} \tag{3.12}$$

just as in the market economy.

4. Uncertainty and Expectations

So far we have been dealing with economies without uncertainty. But the real world is full of uncertainty. In this section, we introduce uncertainty in two-period economies. We will assume that exogenous variables (technology, endowments, preferences, taxes, money supply etc.) can take more than one value in the second period. We will also assume that uncertainty about the values of exogenous variables can be expressed in terms of their probability distributions and all agents in the economy know these distributions. The question we are going to ask is: how allocations and prices are determined in economies in which agents face uncertainty about exogenous variables in the second period (no uncertainty in the first period)? The DGE model with uncertainty is known as **Dynamic Stochastic General Equilibrium (DSGE) model**.

Example 10

We begin with an example. Let us modify the environment example 6 by assuming that there is uncertainty about production function in the next period. Let the new production function be Ak^α where A is a random variable which can take values A^h and A^l with probabilities p^h and p^l respectively ($p^h + p^l = 1$). Consider h to be high state and l to be low state in the sense that $A^h > A^l$. We continue to assume that the depreciation rate $\delta = 1$. Now we want to find out allocations and prices in this economy. Before we proceed, let us define the expectation operator E. The expected value of A is given by

$$E(A) = p^h A^h + p^l A^l. \tag{4.1}$$

Notice that there are two states in the second period: high state and low state. Corresponding to these two states, there will be two consumption levels c_2^h and c_2^l in the second period. Thus, in this economy the objects of interest are $c_1,\ k,\ c_2^h,\ c_2^l,\ r$. In order to solve for these variables, let us setup the representative agent problem:

$$\max_{c_1, c_2^h, c_2^l, k} U = \ln c_1 + \beta[p^h \ln c_2^h + p^l \ln c_2^l] \equiv \ln c_1 + \beta E \ln c_2 \tag{4.2}$$

subject to

$$c_1 + k = y \tag{4.3}$$

$$c_2^h = A^h k^\alpha \tag{4.4}$$

$$c_2^l = A^l k^\alpha. \tag{4.5}$$

We have two constraints on the second period consumption corresponding to two states. Ultimately, only one of these will end up binding. Putting the budget constraints in the objective function, we have

$$\max_k U = \ln(y - k) + \beta[p^h \ln(A^h k^\alpha) + p^l \ln(A^l k^\alpha)] \equiv \ln c_1 + \beta E \ln c_2 \tag{4.6}$$

The first order condition is given by

$$\frac{1}{y-k} = \frac{\alpha\beta}{k}. \tag{4.7}$$

(4.7) is an example of the **Euler equation**. The solution for optimal k is

$$k = \frac{\alpha\beta}{1+\alpha\beta} y \tag{4.8}$$

The optimal consumption plan is given by

$$c_1 = \frac{1}{1+\alpha\beta} y, \; c_2^i = A^i \left[\frac{\alpha\beta}{1+\alpha\beta} y \right]^\alpha \text{ for } i = h, l. \tag{4.9}$$

Now we have solved for optimal allocations. We can solve for the real rate of interests by using MPK. The real rate of interest will satisfy

$$r = E(A\alpha k^{\alpha-1}) - 1. \tag{4.10}$$

We have characterized allocations and prices for this particular example. Let us do it for a more general case.

Example 11

Suppose that the period utility is $u(c)$ with $u'(c) > 0$ and $u''(c) < 0$. The production function is $y = Af(k)$ with $f(0) = 0$, $f'(k) > 0$ and $f''(k) < 0$. Suppose that $f(k)$ satisfies Inada Conditions: $f'(0) = \infty$ and $f'(\infty) = 0$. As before $\delta = 1$.

The representative agent problem is

$$\max_{c_1, c_2^h, c_2^l, k} U = u(c_1) + \beta[p^h u(c_2^h) + p^l u(c_2^l)] \equiv u(c_1) + \beta E(u(c_2)) \tag{4.11}$$

subject to

$$c_1 + k = y \tag{4.12}$$

$$c_2^h = A^h f(k) \tag{4.13}$$

$$c_2^l = A^l f(k). \tag{4.14}$$

We can plug these constraints in the objective function (4.11) and get unconstrained maximization problem:

$$\max_k U = u(y-k)+\beta[p^h u(A^h f(k))+p^l u(A^l f(k))] \equiv u(y-k)+\beta E(u(Af(k))). \tag{4.15}$$

The first order condition satisfies

$$u'(c_1) = \beta \left[p^h u'(c_2^h)A^h f'(k) + p^l u'(c_2^l)A^l f'(k)\right]. \tag{4.16}$$

Using the expectation operator defined in (4.1), we can write (4.16) as

$$u'(c_1) = \beta E\left[u'(c_2)Af'(k)\right]. \tag{4.17}$$

Equations like (4.17) are known as **Euler equation**. It has straight forward interpretation. At the optimal level of k, the marginal cost of k (LHS) equals the expected marginal benefit from k (RHS). The marginal cost of investment is simply equal to the marginal utility of consumption forgone in the current period $u'(c)$. What is the gain from one unit of investment? One unit of investment produces $Af'(k)$ units of goods next period. In terms of utility this benefit is simply equal to $u'(c_2)Af'(k)$. Since, this utility occurs next period, we need to discount it in order to make it comparable to the current utility, and thus the expected marginal benefit from investment is given by the RHS of (4.17). Using this Euler equation together with the resource constraints we can derive optimal allocations. Once we get optimal allocations, using MPK we can get the real rates of interest.

Exercise: Show that the production function $y = Ak^\alpha$ satisfies Inada conditions. Let $u(c) = \ln c$. Using the Euler equation (4.17) and resource constraints show that optimal allocations and prices satisfy (4.8), (4.9), and (4.10).

5. Precautionary Saving

Income Risk and Precautionary Saving, Capital Income Risk

The risk-averse consumer's problem is

$$\max_{c_1,c_2,s} U(c_1) + \beta U(c_2)$$

subject to

$$c_1 + s = y_1 \ \& \tag{1}$$

$$c_2 = \overline{R}s + \overline{y}_2 \tag{2}$$

where y_1, $\overline{y}_2$ & $\overline{R}$ are first period income, second period income, and the rate of interest respectively. Let $y_1 > \overline{y}_2$.

The optimal amount of saving is given by

$$U_c(y_1 - s) = \beta\overline{R}U_c(\overline{R}s + \overline{y}_2) \tag{3}$$

Denote the optimal amount of saving by $\overline{s}$.

Income Risk and Saving

Now suppose that the second period income, y_2, of the consumer is uncertain. The consumer has to save in the first period before knowing its second period income. Suppose that

$$Ey_2 = \overline{y}_2 \tag{4}$$

where E is the expectation operator. Denote the optimal amount of saving with uncertain income by s^*. The question is whether $s^* > \overline{s}$. Will uncertainty about future income will lead to higher saving (**precautionary saving**) in the first period?

Now the consumer's problem is

$$\max_{c_1,c_2,s} U(c_1) + \beta EU(c_2)$$

subject to

$$c_1 + s = y_1 \text{ \&} \tag{5}$$

$$c_2 = \overline{R}s + y_2. \tag{6}$$

The optimal amount of saving, s^*, is given by

$$U_c(y_1 - s) = \beta\overline{R}EU_c(\overline{R}s + y_2). \tag{7}$$

Now compare (3) and (7). The LHS of both equations has identical expressions. However, the RHS has different expressions. Given the concavity of the utility function (or diminishing marginal utility), it is immediately clear that if

$$EU_c(\overline{R}s + y_2) > U_c(\overline{R}s + \overline{y}_2) \equiv U_c(E(\overline{R}s + y_2)) \tag{8}$$

$s^* > \overline{s}$.

It is the mathematical property (**Jensen's inequality**) that if a function $f(x)$ is a convex function of the random variable x then

$$Ef(x) > f(E(x)) \equiv f(\overline{x}) \tag{9}$$

where $\overline{x} = E(x)$.

Essentially we need to find out under what condition $U_c(\overline{R}s+y_2) \equiv U_c(c_2)$ is a convex function of y_2. Taking the second-derivative of $U_c(c_2)$ w.r.t. y_2 you can see that when the third derivative of the utility function is positive i.e. $U_{ccc}(c) > 0$, $U_c(c_2)$ will be a convex function of y_2. In this case, $s^* > \overline{s}$. The difference between $s^* - \overline{s}$ is known as *precautionary saving*. This is extra saving due to uncertainty in the future income.

Example 1: Suppose that $U(c) = \frac{c^{1-\mu}}{1-\mu}$. In this case, $U_{ccc}(c) = \mu(1+\mu)c^{-(\mu+1)} > 0$. Thus, in case of income uncertainty there will be extra-saving.

Example 2: Suppose that we have quadratic utility function, $U(c) = ac - bc^2$. Since $U_{ccc}(c) = 0$, in this case, there will be no extra saving due to income uncertainty. Thus, the mere fact that an individual is **risk-averse** does not mean there will be precautionary saving. This is known as **certainty-equivalence** result.

Capital Income risk

Now suppose that y_2 is certain, but there is uncertainty about the rate of interest R. Let R be the random variable with $E(R) = \overline{R}$. Now the optimization problem is

$$\max_{c_1,c_2,s} U(c_1) + \beta EU(c_2)$$

subject to

$$c_1 + s = y_1 \ \& \tag{10}$$

$$c_2 = Rs + \overline{y}_2 \tag{11}$$

The optimal amount of saving, s^{**}, is given by

$$U_c(y_1 - s) = \beta EU_c(Rs + \overline{y}_2)R. \tag{12}$$

We want to know under what condition $s^{**} > \overline{s}$. As before we need to derive the condition under which $U_c(Rs + \overline{y}_2)R \equiv U_c(c_2)R$ is a convex function of R. Taking the second derivative, you can show that $U_c(c_2)R$ is a convex function of R if

$$U_{ccc}(c_2)Rs + 2U_{cc}(c_2) > 0 \text{ or} \tag{13}$$

$$-\frac{U_{ccc}(c_2)}{U_{cc}(c_2)}Rs > 2. \tag{14}$$

As is evident $U_{ccc}(c_2) > 0$ is no longer sufficient (though necessary) to ensure extra-saving.

For example, with $U(c) = \frac{c^{1-\mu}}{1-\mu}$, (14) is satisfied only when $(\mu + 1)\frac{Rs}{c_2} > 2$. The reason is that risky capital income affects saving in two opposite ways. Firstly, the precautionary motive ($U_{ccc}(c_2) > 0$) has positive effect on saving. However, risky capital income also reduces the attractiveness of saving (negative substitution effect), which is captured by the term $U_{cc}(c_2)$. Only if the precautionary motive dominates the negative substitution effect, saving will be higher than the certainty case.

6. Government Expenditure and Ricardian Equivalence

So far, we have considered economies without government expenditure. Suppose that there is a government which consumes quantity g_1 and g_2 in period 1 and 2 respectively. It can finance its expenditure through lump-sump taxation and issuing one period bond. Let b_1 be the bond issued in period 1 and let t_1 and t_2 be lump sum taxes in the first and the second period respectively. Let r be the net rate of interest.

The period budget constraints of the government are:

$$g_1 = t_1 + b_1 \ \& \tag{6.1}$$

$$g_2 = t_2 - (1 + r)b_1. \tag{6.2}$$

Combining (6.1) and (6.2), we can write inter-temporal budget constraint of the government:

$$t_1 + \frac{t_2}{1 + r} = g_1 + \frac{g_2}{1 + r}. \tag{6.3}$$

The left hand side of (5.3) is the present value of revenue of the government and the right hand side is the present value of expenditure.

Let us now consider how government expenditure and financing of its expenditure affects decisions of the private agents and market allocations. Let $u(c_1, c_2)$ be the utility function of the representative consumer. Also suppose that the representative consumer has endowments y_1, y_2. Now the period budget constraints for the representative consumer are:

$$c_1 = y_1 - b_1 - t_1 \text{ \&} \tag{6.4}$$

$$c_2 = y_2 - t_2 + (1+r)b_1. \tag{6.5}$$

Combining (6.4) and (6.5), we can write inter-temporal budget constraint of the representative consumer:

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} - t_1 - \frac{t_2}{1+r} \equiv W \tag{6.6}$$

where W is the present value of his net life-time income or wealth. The left hand side of (6.6) is the present value of expenditure of the representative consumer. The representative consumer chooses c_1 and c_2 to maximize his utility subject to (6.6). Using the maximization problem one can derive c_1 and c_2 as a function of W.

Note that W is just a function of taxes and endowments. It is independent of bonds, b_1. Thus changes in b_1 or deficit financing does not affect net wealth of the representative consumer. In that sense, the government bonds are not net wealth – they signal deferred taxes and do not affect lifetime budget constraints in the private sector. If we combine (6.3) and (6.6), we have

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} - g_1 - \frac{g_2}{1+r} \equiv W. \tag{6.7}$$

(6.7) shows that in equilibrium W depends on endowments and government expenditure. In essence, in this economy the representative consumer maximizes his utility subject to (6.7). Immediate consequence is that the mode of financing government expenditure (whether through taxes or government bonds) does not affect market allocations and prices. Timing of taxes and government budget deficits do not affect market allocations and prices. In that sense, financing government expenditure through either taxes or budget deficit is equivalent.

The result that taxes and budget deficit are equivalent ways of financing government expenditure is known as the **Ricardian Equivalence**. Note that Ricardian equivalence does not say that government expenditure does not affect market allocations and prices. From (6.7) it is clear that any change in government expenditure affects wealth of representative consumer and thus market allocations and prices.

The Ricardian equivalence result is quite striking. However, it depends on a number of special assumptions such as : (i) no uncertainty (ii) lump-sum taxes (iii) no heterogeneity and (iv) perfect capital market. If we relax these assumptions, in general, Ricardian equivalence will not hold and timing of taxes and government budget deficits would matter.