Practice Questions for Mid-Term I

**Question 1:** Consider the Cobb-Douglas production function in intensive form:

\[ y = f(k) = k^\alpha; \quad \alpha \in (0,1) \quad (1) \]

where \( y \) and \( k \) are output per worker and capital per worker respectively. Suppose that the labor force growth rate is \( n \) and there is no depreciation and technical growth. Let \( s \) be the savings rate.

a. Show that if \( s = \alpha \), then the competitive equilibrium and the golden-rule equilibrium coincide on the balanced growth path.

b. Suppose that \( s = 2\alpha \). Is the competitive equilibrium capital stock per worker is higher or lower than the golden rule equilibrium on the balanced growth path?

**Suggested Answer:** Balanced growth path is characterized by:

\[ k^{\alpha-1} = \frac{n}{s} \quad (1) \]

which implies

\[ k^* = \left( \frac{s}{n} \right)^{\frac{1}{1-\alpha}}. \quad (2) \]

The golden rule is characterized by

\[ \alpha k^{\alpha-1} = n \quad (3) \]

which implies

\[ k_g = \left( \frac{\alpha}{n} \right)^{\frac{1}{1-\alpha}}. \quad (4) \]

Using (2) and (4), you can show that \( k^* = k_g \) when \( s = \alpha \) and \( k^* > k_g \) when \( s = 2\alpha \).

**Question 2:** Now modify question 1 as follows. Suppose that there is labor augmenting technical growth. Technology grows at a constant rate \( \gamma \), i.e \( E(t) = E(0) \exp^{\gamma t} \). Suppose that the capital stock depreciates at a constant rate \( \delta \). Let \( y \) and \( k \) denote output and capital stock per effective-labor.

a. Derive the expressions for output and capital per worker on the balanced growth path.

b. Derive the expressions for the wage rate and profit rate on the balanced growth path.

**Suggested Answer:** On the balanced growth path

\[ k^* = \left( \frac{s}{n + \gamma + \delta} \right)^{\frac{1}{1-\alpha}}. \quad (1) \]

Then
\[ \frac{K}{L} = \left( \frac{s}{n + \gamma + \delta} \right)^{\frac{1}{1-\alpha}} E_0 \exp^{\gamma t} \& \] 
(2)

\[ \frac{Y}{L} = \left( \frac{s}{n + \gamma + \delta} \right)^{\frac{\alpha}{1-\alpha}} E_0 \exp^{\gamma t}. \] 
(3)

For wage rate and profit rate, wage is simply the marginal product of labor and profit rate is the marginal product of capital.

**Question 3.** Consider a Solow model with no technical progress. Suppose that savings rate, \( s = 0.12 \), the depreciation rate is \( \delta = 0.04 \) and the population growth rate is \( \eta = 0.02 \). The production function in the intensive form is

\[ f(k) = k^{\alpha} \]

where \( k \) is capital-labor ratio and \( \alpha = 0.4 \).

(i) Find the real rate of interest and real wage rate in the steady state equilibrium.

Suggested Answer:

\[ k = \left( \frac{s}{n + \delta} \right)^{\frac{1}{1-\alpha}}. \]

which implies \( k^* = 3.17 \), \( r = \alpha k^{\alpha-1} - \delta = 0.157 \), \( w = (1 - \alpha)k^\alpha = 0.96 \).

(ii) Derive the golden rule level of capital-labor ratio. Is the savings above or below the golden rule level?

\[ k^g = \left( \frac{\alpha}{n + \delta} \right)^{\frac{1}{1-\alpha}} \]

which implies \( k^g = 23.76 \). Thus, \( k^g > k^* \).

(iii) Describe qualitatively the effects on output and output per-capita of a transition to a higher savings rate.

Suggested Answer: Higher \( s \) increases \( k \) and during transition capital stock out grow faster than the labor force growth rate.

**Question 4.** Consider a Solow model with technical progress. Let the production function be of the form

\[ Y = F(K, EL) \]

where \( K \) and \( EL \) are capital stock and the effective labor force respectively. Production function is strictly increasing and concave function of \( K \) and \( EL \). Assume that production function has constant returns to scale. Suppose that the effective labor \( EL = E^*L \). Let \( n \) and \( \delta \) be the labor force growth rate and the depreciation rate respectively. Let \( \lambda \) be the rate of labor-augmenting technical growth, i.e. \( E(t) = E(0) \exp^{\lambda t} \). Suppose that capital and labor are paid their marginal products. In addition, all capital income is saved and all labor income is consumed. Thus,
\[
\dot{K} = MPK \cdot K - \delta K
\]
where \( MPK \) is the marginal product of capital.

(i) Denote the effective capital-labor ratio by \( k(= K/EL) \). Derive the differential equation characterizing the time path of \( k \).

(ii) Derive the equation which characterizes the steady state values of \( k \). Let \( Y = K^\alpha EL^{1-\alpha} \) with \( 0 < \alpha < 1 \). Using a diagram show that this economy will converge to a balanced growth path for any initial value of \( k \geq 0 \). Suppose that the initial value of \( k > 0 \). Discuss the process by which the economy would reach its balanced growth path.

(iii) Derive the expression for steady state value of \( k(>0) \), wage rate, \( w \), and the real rate of interest, \( r \).

Suggested Answer:

By taking the derivative of \( k \) w.r.t. time and using the expression for \( \dot{K} \) above you can derive

\[
\dot{k} = MPK \cdot k - (\delta + n + \lambda)k = (MPK - (\delta + n + \lambda))k. \tag{1}
\]

The steady state is characterized by

\[
(MPK - (\delta + n + \lambda))k = 0. \tag{2}
\]

Two steady states \( k_1^* = 0 \) and \( k_2^* > 0 \). \( k_2^* \) will be stable steady state and given by

\[
MPK \equiv \alpha k^{\alpha-1} = n + \delta + \lambda. \tag{3}
\]

You can draw the standard diagram, one depicting \((\delta + n + \lambda)k\) and other \( MPK \cdot k \). Alternatively, you can draw the phase diagram. For any initial value of \( k > 0 \) the economy will converge to \( k_2^* \). If \( k < k_2^* \), then \( MPK > \delta + n + \lambda \) and the capital investment will be larger than the break-even point increasing the capital stock. This process will go on till the economy reaches \( k_2^* \). Opposite will be the case if \( k > k_2^* \).

With \( Y = K^\alpha EL^{1-\alpha} \),

\[
k_2^* = \left( \frac{\alpha}{\delta + n + \lambda} \right)^{\frac{1}{1-\alpha}}.
\]

\[
w = MPL = (1 - \alpha)\left( \frac{\alpha}{\delta + n + \lambda} \right)^{\frac{\alpha}{1-\alpha}} E(0) \exp^{\lambda t}.
\]

\[r = MPK - \delta = n + \lambda.\]

**Question 5.** Suppose that the representative consumer has endowment of good \( y \) in the first period and endowment of labor in the second period. Also suppose that the second
period production function is $k^{\alpha}n^{1-\alpha}$, where $k$ is the investment made in the first period. The depreciation rate is 100%. The objective function of the consumer is

$$\ln c_1 + \beta \ln c_2 + \beta \ln(1 - n)$$

where $n$ is the labor supplied in the second period. Define competitive equilibrium and find out equilibrium allocations and prices.

**Suggested Answer:** The budget constraints are

$$c_1 + k = y & (1)$$
$$c_2 = k^{\alpha}n^{1-\alpha}. & (2)$$

Put (1) and (2) in the objective function. Then by maximizing the objective function, you can show that

$$k = \frac{\alpha \beta y}{1 + \alpha \beta} \text{ & } n = \frac{1 - \alpha}{2 - \alpha}. \quad (3)$$

The prices are given given by

$$MPK = 1 + r, \text{ & } MPN = w. \quad (4)$$

Put expressions for $k, n$ derived in (3) in (4).

The competitive equilibrium is constituted by allocations $c_1, c_2, k, n$ and prices $r, w$, such that a) given prices consumers maximize their utility subject to their budget constraint and firms maximize their profits, b) markets clear: $c_1 + k = y$, $c_2 = k^{\alpha}n^{1-\alpha}$ and $n^d = n^s \ ie. \ labor \ demand \ equals \ labor \ supply.$

**Question 6.** There are two time periods. The utility function is

$$(a - bc_1^2) + E\frac{1}{1 + r}(a - bc_2^2)$$

where $E$ is the expectation operator. The budget constraint is

$$c_1 + \frac{E}{1 + r}c_2 = y_1(1 - t_1) + E\frac{y_2(1 - t_2)}{1 + r}$$

where $y_i$ denotes labor income and $t_i$ denotes tax rate for $i = 1, 2$. Suppose that there is a linear technology in this economy, which determines the constant interest rate $r$.

(a.) Solve for $c_1$ and $E(c_2)$.

(b.) Suppose that $y$ and $r$ are constant. In period 1 the tax authority announces the following rule for setting taxes in period 2:

$$t_2 = \lambda t_1 + \epsilon_2$$
where $\epsilon_2$ is an unpredictable shock. Let $E(\epsilon_2) = 0$. Find the statistical relationship between $c_1$ and $t_1$.

(c.) In general, can we evaluate the effects of tax changes using decision rules? Hint: Lucas Critique.

**Suggested Answer:** Again by setting the utility maximization problem you can show that

$$c_1 = E(c_2).$$

By putting (1) in the budget constraint you can show that

$$c_1 = E(c_2) = \frac{1 + r}{2 + r} \left[ y_1 (1 - t_1) + \frac{y_2 (1 - t_2)}{1 + r} \right].$$

By putting the policy rule in (2), you can show that

$$c_1 = \frac{1 + r}{2 + r} \left[ y_1 + \frac{y_2}{1 + r} - \left[ y_1 + \frac{\lambda}{1 + r} \right] t_1 \right].$$

In order to find out the relationship between consumption and tax normally, we regress consumption on tax rate. In this case, the estimated co-efficient of $t_1$, $B \equiv \frac{1 + r}{2 + r} (y_1 + \frac{\lambda}{1 + r})$. In order to find out, the effect of tax changes we simulate the model for various values of tax rate. However, this procedure is valid only when the coefficient of tax rate $B$ is constant or is invariant to changes in tax rates/ policy/ regime. However, $B$ depends on $\lambda$ and if there is tax changes, there might be changes in $\lambda$ too and thus the original estimate of $B$ may not be valid under new tax policy/ regime. This may lead to incorrect estimate of the effects of taxes. This is the essence of Lucas critique. With the change in policies/regimes, the behavioral relationships change (summarized in the coefficients like B). Historically observed relationships may not be correct guide to the future if policies change. Thus, using decision rules to gauge effects of policies can be problematic.

**Question 7. Government Expenditure:** Suppose that there is a government which consumes quantity $\{g_1, g_2\}$. It can finance its expenditure through through imposing lump-sum taxes and issuing bonds. The representative consumer has endowments $\{y_1, y_2\}$. The utility function of the representative consumer is

$$\ln c_1 + \beta \ln c_2.$$

(a.) Write down the inter-temporal budget constraints of the government and the representative consumer.

**Suggested Answer:** The intertemporal budget constraints for the govt. and the consumers are

$$g_1 + \frac{g_2}{1 + r} = t_1 + \frac{t_2}{1 + r}$$

(1)
\[ c_1 + \frac{c_2}{1 + r} = y_1 + \frac{y_2}{1 + r} - t_1 - \frac{t_2}{1 + r} \equiv W. \]  
(2)

(b.) Solve for \( c_1, c_2 \) and \( r \).

Suggested Answer: The first order condition implies that

\[ \frac{c_2}{1 + r} = \beta c_1. \]  
(3)

From (2) and (3), we have

\[ c_1 = \frac{1}{1 + \beta} W \]  
(4)
\[ c_2 = \frac{\beta(1 + r)}{1 + \beta} W. \]  
(5)

(c.) Define the competitive equilibrium: Definition is standard. The market clearing conditions will be

\[ c_1 + g_1 = y_1 \]  
(6)
\[ c_2 + g_2 = y_2. \]  
(7)

(6) and (7) pin down \( c_1 \) and \( c_2 \). Then from (3) we have

\[ 1 + r = \frac{y_2 - g_2}{\beta(y_1 - g_1)}. \]  
(8)

(d.) Does the Ricardian Equivalence hold in this economy?

Suggested Answer: In this economy, the Ricardian equivalence holds. The levels of \( t_1, t_2 \) and \( b \) (borrowing) will be indeterminate. For example, if \( b = 0 \) then \( t_1 = g_1 \) and \( t_2 = g_2 \).

At the other extreme, if \( g_1 = b \), then \( t_2 = g_2 + b(1 + r) = g_2 + g_1(1 + r) = g_2 + g_1 \frac{y_2 - g_2}{\beta(y_1 - g_1)}. \)

Anything in between these two extremes is equally fine.

**Question 8. Borrowing and Lending**

Consider a two-period economy with competitive markets. Suppose that there are two types of individuals (equal numbers): borrowers \((b)\), with no endowment in the first period and endowment \( y \) in the second period, and lenders \((l)\) with endowment \( y \) in the first period and no endowment in the second period. Both borrowers and lenders have logarithmic utility function:

\[ \ln c_{1,i} + \beta \ln c_{2,i}, \forall i = b, l \]
where 0 < β < 1 is the discount rate and \( c_{1,i} \) and \( c_{2,i} \) are consumption of the ith type of individual in period 1 and 2 respectively. Let \( r \) be the market rate of interest.

i. Derive the first order conditions characterizing the optimal choices of a lender and interpret them.

ii. What is the effect of changes in the market interest rate on the optimal choice of lending? Provide the intuition.

iii. Derive the first order conditions characterizing the optimal choices of a borrower and interpret them.

iv. What is the effect of changes in the market interest rate on the optimal choice of borrowing? Provide the intuition.

v. Define the competitive equilibrium.

vi. Derive the equilibrium market rate of interest and allocations. What is the effect of a change in \( \beta \) and \( y \) on the equilibrium lending/borrowing and the market rate of interest? Provide the intuition.

Suggested Answer:

Let \( l \) and \( b \) denote the lending and borrowing respectively in the first period. By setting up the lender’s problem, you can show that the foc is given by

\[
\frac{1}{y-l} = \frac{\beta}{l}.
\]

The LHS is MC of lending and the RHS is the MB. One unit extra lending reduces lender’s utility by \( \frac{1}{c_{1,l}} = \frac{1}{y-l} \). At the same time lender’s utility increases next period by \( \frac{\beta}{c_{2,l}(1+r)} = \frac{\beta}{l} \). Using the foc you can solve for optimal lending

\[
l = \frac{\beta}{1 + \beta} y.
\]

Notice that the amount of lending is independent of \( r \). Reason is that in the logarithmic utility case, the income and the sub effects of changes in \( r \) cancel each other out.

By setting up the borrower’s problem, you can show that the foc is given by

\[
\frac{1}{b} = \frac{\beta(1+r)}{y - (1+r)b}.
\]

The LHS is MB of borrowing and the RHS is the MC. One unit extra borrowing increases borrower’s utility by \( \frac{1}{c_{1,b}} = \frac{1}{b} \). At the same time borrowers’s utility falls next period by \( \frac{\beta}{c_{2,b}(1+r)} = \frac{\beta(1+r)}{y - (1+r)b} \). Using the foc you can solve for optimal borrowing

\[
b = \frac{y}{(1 + \beta)(1 + r)}.
\]
Notice that the amount of borrowing falls as \( r \) rises. Reason is that for the borrower, the income and the sub effects of changes in \( r \) reinforce each other.

**Definition of The Equilibrium**

**Competitive Equilibrium:** A competitive equilibrium is the price (real rate of interest) \( r \) and allocations \( \{c_{1,l}, c_{2,l}, c_{1,b}, c_{2,b}, b, \& \ l\} \) such that:

(a) the representative lender maximizes its utility given prices and subject to its budget constraints;

(b) the representative borrower maximizes its utility given prices and subject to its budget constraints; and

(c) markets clear:

\[
\begin{align*}
c_{1,l} + c_{1,b} &= y, & c_{2,l} + c_{2,b} &= y, & l(r) = b(r).
\end{align*}
\]

Using the expressions above and the definition of equilibrium you can derive the allocations and the market rate of interest \( 1 + r = \frac{1}{\beta} \). This suggests that an increase in \( \beta \) reduces the market rate of interest. The reason is that as \( \beta \) rises, individuals become more patient and discount future less heavily i.e. future consumption becomes less costly relative to the present consumption. This also implies that the lender will be willing to lend more (increased supply), for a given \( r \). Thus the amount of equilibrium lending and borrowing will increase and \( r \) will fall. An increase in \( y \) reduces the marginal cost of both lenders and borrowers leading to more equilibrium borrowing and lending.