Math Supplement (Derivatives and Optimization)

In this supplement, we very briefly review the mathematical techniques and methods used in this course. Denote a function of one variable: $y = f(x)$.

Derivative: measures the incremental change in function y for a given change in x. The derivative of a function can be denoted in many ways $\frac{dy}{dx} \equiv \frac{df(x)}{dx} \equiv f_x$. Some important rules of taking derivatives (differentiation) are:

$$y = k \text{ (constant) then } \frac{dy}{dx} = 0.$$

$$y = x^n \text{ then} \frac{dy}{dx} = nx^{n-1}.$$

$$y = \ln x \text{ then} \frac{dy}{dx} = \frac{1}{x}.$$

$$y = \exp^x \text{ then} \frac{dy}{dx} = \exp^x.$$

Partial Derivative

Denote a function of two variables: $y = f(x_1, x_2)$.

Partial derivative – measures the rate of change of the function y wrt (with respect to) one variable holding other variables constant. Suppose that

$$y = x_1^2 x_2^2. \tag{1}$$

Then the partial derivative of y wrt x_1 is given by

$$\frac{\partial y}{\partial x_1} = 2x_1 x_2^2. \tag{2}$$

Partial Derivative

The partial derivative of y wrt x_2 is given by

$$\frac{\partial y}{\partial x_2} = 2x_1^2 x_2. \tag{3}$$

There are several ways to denote the partial derivatives:

$$\frac{\partial y}{\partial x_i} \equiv \frac{\partial f(x_1, x_2)}{\partial x_i} \equiv f_{x_i} \equiv f_i \; \forall i = 1, 2 \tag{4}$$

Exercises: Derive the partial derivatives of $\ln(x_1 + x_2)$, $\ln(x_1) + \ln(x_2)$, and $x_1^{\alpha} x_2^{1-\alpha}$.

Total Derivative

Total derivative – measures the total incremental change in the function when all variables are allowed to change:

$$dy = f_1 dx_1 + f_2 dx_2. \tag{5}$$

Let $y = x_1^2 x_2^2$. Then the total derivative of function y is given by

$$dy = 2x_1 x_2^2 dx_1 + 2x_1^2 x_2 dx_2. \tag{6}$$

Note that the rules of partial and total derivative apply to functions of more than two variables.

Exercises: Derive the total derivative of $\ln(x_1 + x_2)$, $\ln(x_1) + \ln(x_2)$, and $x_1^\alpha x_2^{1-\alpha}$.

Some Further Rules of Differentiation

$$\text{If } y = f(x) \pm g(x) \text{ then } \frac{dy}{dx} = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}. \tag{7}$$

Example: If $y = x^2 + 2x$ then $\frac{dy}{dx} = 2x + 2$.

$$\text{If } y = f(x)g(x) \text{ then } \frac{dy}{dx} = \frac{df(x)}{dx}g(x) + \frac{dg(x)}{dx}f(x). \tag{8}$$

Example: If $y = x(x^2 + 1)$ then $\frac{dy}{dx} = 3x^2 + 1$.

$$\text{If } y = \frac{f(x)}{g(x)} \text{ then } \frac{dy}{dx} = \frac{1}{g^2(x)}\left[\frac{df(x)}{dx}g(x) - \frac{dg(x)}{dx}f(x)\right]. \tag{9}$$

Example: If $y = x/1 + x$ then $\frac{dy}{dx} = \frac{1}{(1+x)^2}$.

Chain Rule: Let $z = f(y)$ and $y = g(x)$ then

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}. \tag{10}$$

Example: If $z = \ln(1 + 2x)$ then $\frac{dz}{dx} = \frac{2}{1+2x}$.

Optimization

Optimization– involves finding the maximum or minimum (or stationary) value of function. It involves taking the derivative of a function and setting it to zero.

Suppose that $y = f(x)$. Then to derive maximum or minimum value all we need to do is to derive $\frac{dy}{dx}$ and then set $\frac{dy}{dx} = 0$. $\frac{dy}{dx}$ is called the first derivative. The condition that $\frac{dy}{dx} = 0$ is called the first-order condition.

To know whether the solution of the function $\frac{dy}{dx} = 0$ gives maximum or minimum, we need to take the derivative of the first derivative, $\frac{dy}{dx}$. The derivative of the first derivative is called the second derivative. Let $\frac{dy}{dx} \equiv z$, then the solution of $\frac{dy}{dx} = 0$ is a maximum if $\frac{dz}{dx} \equiv \frac{d^2y}{dx^2} < 0$. If $\frac{dz}{dx} \equiv \frac{d^2y}{dx^2} > 0$, then we have got a minimum.

Optimization

An example: Let $y = \ln x - 2x$. Find x at which function y is maximized.
$\frac{dy}{dx} = \frac{1}{x} - 2$. Setting $\frac{dy}{dx} = \frac{1}{x} - 2 = 0$, we get $x = \frac{1}{2}$.
Exercise: Why at $x = \frac{1}{2}$, the function y is maximized.
Exercise: Find the value of x at which $y = x^2 - 2x$ is minimized.

In this course, we will not be bothered about the sign of the second derivative. If question asks about the maximum, then the function given would be such as to have negative second derivative. Similarly, in the case of minimum the function would have positive second derivative.

Optimization of a Function with Two Variables

Let $y = f(x_1, x_2)$. To find maximum or minimum, we take first derivative of y wrt to both x_1 and x_2 and then set them equal to zero *ie*., derive first-order conditions.

$$\frac{\partial y}{\partial x_1} = 0. \tag{11}$$

$$\frac{\partial y}{\partial x_2} = 0. \tag{12}$$

(11) and (12) yield two equations in two unknowns x_1 and x_2. The solution of these two equations gives us the values of x_1 and x_2 at which y attains maximum or minimum.

Optimization of a Function with Two Variables

An Example: Let $y = x_1^2 + 2x_2^2 + x_1 x_2$. We want to find values of x_1 and x_2 at which y attains the minimum. The first order conditions are

$$\frac{\partial y}{\partial x_1} = 2x_1 + x_2 = 0. \tag{13}$$

$$\frac{\partial y}{\partial x_2} = 4x_2 + x_1 = 0. \tag{14}$$

Combining (13) and (14), we get $x_1 = 0$ and $x_2 = 0$.

Constrained Optimization

Many times in economics, we face the optimization problem involving constraints e.g. utility maximization by a consumer subject to his/her budget constraint. We can solve this kind of problem using the technique discussed earlier.

An example: Suppose that the utility function of a consumer is $U(c_1, c_2)$, where c_1 and c_2 are consumption of good 1 and good 2 respectively. Suppose that the budget constraint faced by the consumer is $p_1 c_1 + p_2 c_2 = I$, where p_1, p_2, and I are prices of good 1 and good 2 and income respectively. We want to find out the optimal choices of c_1 and c_2 (consumption bundle) given prices and income.

Constrained Optimization

The consumer's problem is

$$\max_{c_1,c_2} U(c_1, c_2) \tag{15}$$

subject to

$$p_1 c_1 + p_2 c_2 = I. \tag{16}$$

The easiest way to solve this problem is to put the budget constraint (equation 16) in the utility function (or the objective function). This way we convert the constrained optimization problem in an unconstrained optimization problem. Combining (15) and (16) we have

$$\max_{c_1} U(c_1, \frac{I}{p_2} - \frac{p_1}{p_2} c_1). \tag{17}$$

Constrained Optimization

The first order condition is

$$U_1 = \frac{p_1}{p_2} U_2. \tag{18}$$

In deriving (18) we use the chain rule. (18) can be rewritten as

$$\frac{U_1}{U_2} = \frac{p_1}{p_2}. \tag{19}$$

(19) equates the marginal rate substitution to the ratio of prices. Using equations (19) and (16) we can derive the optimal values of c_1 and c_2.

Constrained Optimization: An Example

Question: Let $U(c_1, c_2) = \ln c_1 + \ln c_2$. Derive the optimal consumption bundle given the budget constraint (eq. 16).

From (19) we have

$$p_1 c_1 = p_2 c_2. \tag{20}$$

Putting (20) in the budget constraint we have $c_1 = \frac{1}{2}\frac{I}{p_1}$. Then (20) implies $c_2 = \frac{1}{2}\frac{I}{p_2}$.

Two Periods: Consumption-Savings Choice

Suppose that a consumer lives for two periods. The utility function of the consumer is $U(c_1, c_2)$, where c_1 and c_2 are consumption in period 1 and period 2 respectively. Suppose that consumer has income Y in the first period, but has no income in the second period. Consumer has to save in the first period in order to consume in the second period. Let s be the savings in the first period and r be the rate of interest.

We want to find out the optimal choices of c_1, c_2 (consumption bundle), and savings, s, for a given rate of interest and income.

Two Periods: Consumption-Savings Choice

The consumer's problem is

$$\max_{c_1, c_2, s} U(c_1, c_2)$$

subject to

$$c_1 + s = Y \; \& \tag{21}$$

$$c_2 = (1 + r)s \tag{22}$$

Two ways to solve the problem

Method 1: Combine the two budget constraints (21 and 22) and derive the inter-temporal budget constraint

$$c_1 + \frac{c_2}{1+r} = Y. \tag{23}$$

Now, the consumer's problem is to

$$\max_{c_1, c_2} U(c_1, c_2)$$

subject to the inter-temporal budget constraint (23).

Two ways to solve the problem

Method 2: Put the budget constraints (21 and 22) in the utility function. The problem becomes

$$\max_{s} U(Y - s, (1 + r)s)$$

In either case, the optimal choices are going to be characterized by following first order condition

$$\frac{U_1}{U_2} = 1 + r. \tag{24}$$

Portfolio-Choice Problem

Now suppose that the consumer can save in terms of two instruments: financial savings (s) and capital investment (k). Capital investment done in period 1 yields output $f(k)$ in the period 2. We want to find out the optimal choices of c_1, c_2 (consumption bundle), and portfolio of savings, s & k, for a given rate of interest and income.

The consumer's problem is

$$\max_{c_1, c_2, s, k} U(c_1, c_2)$$

subject to

$$c_1 + s + k = Y \ \& \tag{25}$$

$$c_2 = (1 + r)s + f(k). \tag{26}$$

The first order conditions imply that

$$f_k(k) = 1 + r. \tag{27}$$

Further Readings

For further studies of these mathematical techniques please consult following books:

Chiang, A. C. and K. Wainwright (2005), “Fundamental Methods of Mathematical Economics”, McGraw-Hill Irwin.
Hoy, M., J. Livernois et. al. (2001), “ Mathematics for Economics”, MIT Press.