Chapter 22
Non-Linear, First-Order Differential Equations

In this chapter, we will learn:

1. How to solve **nonlinear** first-order differential equation?
2. Use of phase diagram in order to understand qualitative behavior of differential equation.

Autonomous Differential Equation

The initial-value problem for an autonomous, nonlinear, first-order differential equation has the following form:

$$\dot{y} = g(y(t)) \ \& \ y(t_0) = y_0 \tag{22.1}$$

where $\frac{dg(y)}{dy}$ & $\frac{d^2g(y)}{dy^2} \neq 0$.

Phase Diagram

Although, it is known that solution to (22.1) exists under the condition that $\frac{dg(y)}{dy}$ is continuous in the neighborhood around t_0, in most cases it is not possible to derive the explicit solution. Often qualitative properties of the differential equation are derived by plotting it. Such plots are known as **phase diagram**.

Steps in Drawing Phase Diagram

Let the differential equation

$$\dot{y} = g(y(t)). \tag{22.2}$$

Our goal is to plot $\dot{y}$ or $g(y(t))$.

Step 1 Take $\dot{y}$ or $g(y(t))$ on y-axis and $y(t)$ on x-axis.

Step 2 Take the first and second derivative of $\dot{y}$ or $g(y(t))$ with respect to y, $\frac{dg(y)}{dy}$ and $\frac{d^2g(y)}{dy^2}$. This gives you the shape of the curve (increasing, decreasing, concave, c-onvex).

Step 3 Derive the steady-state points by setting

$$\dot{y} = g(y(t)) = 0. \tag{22.3}$$

Steady-state or equilibrium points are t-he points at which the curve of $\dot{y}$ or $g(y)$ intersects the x-axis. There can be more than one steady-state point (multiplicity of equilibria).

Given that there can be multiplicity of equilibria, it raises the question which steady-state points are **stable** and which are **unstable**.

Stability Analysis

Stability analysis tells us about the convergence property of the differential equation. A steady state point is stable, if the differential system converges to that point. Otherwise, it is unstable.

> **Theorem 22.2:** A steady-state equilibrium point of a nonlinear first-order differential equation is stable if the derivative $\frac{d\dot{y}}{dy} < 0$ at that point and unstable if the derivative is positive at that point

Solow or Neo-Classical Growth Model

The model relates long run per-worker consumption and growth rate in output to saving rate, work-force growth rate, and technical progress.

Assumptions

1. Constant returns to scale production technology, $Y = F(K, L)$
2. Diminishing Marginal Productivity of Capital
3. Constant Rate of saving (s), thus total savings is $S(t) = sY(t)$,
4. Constant labor force growth rate (n),
5. Constant depreciation rate (δ).
6. No technical progress (Temporary Assumption)

Implications of the Model

1. In the long run economy reaches a stable steady state equilibrium.

2. Per-Worker Consumption (c) in the long run depends on s, n, and δ. There will be no growth in c in the long run.

3. Ultimately, an economy will grow at the rate of work-force growth (n).

Let $I(t)$ be gross investment, then by definition, growth rate of capital stocks is

$$\dot{K} = I(t) - \delta K(t).$$

Since in equilibrium $S(t) = I(t)$, we have

$$\dot{K} = sY(t) - \delta K(t).$$

Now define capital-labor ratio as, $\kappa = \frac{K}{L}$. Then, given constant returns to scale, per-worker output, $y(t)$ can be written as

$$y(t) = \frac{Y(t)}{L(t)} = F(\frac{K(t)}{L(t)}, 1) = f(\kappa(t)).$$

Also

$$\frac{\dot{K}}{L(t)} = \frac{sY(t) - \delta K(t)}{L(t)} = sy(t) - \delta\kappa(t).$$

$$\dot{\kappa} = \frac{\dot{K}}{L(t)} - \kappa\frac{\dot{L}}{L(t)}$$

Combining the above two equations, we get differential equation in the capital-labor ratio wh- ich characterizes Solow growth model.

$$\dot{\kappa} = sf(\kappa(t)) - (\delta + n)\kappa(t).$$

Nonautonomous and Nonlinear Equation

The general form of the nonautonomous, first-order differential equation is

$$\dot{y} = f(t, y). \tag{22.5}$$

The equation can be a nonlinear function of both y and t. We will consider two classes of such equations for which solutions can be easily found: **Bernoulli's Equation** and **Separable Equations**.

Bernoulli's Equation

The differential equation

$$\dot{y} + a(t)y = b(t)y^n, \; n \neq 0 \text{ or } 1 \tag{22.6}$$

is known as Bernoulli's Equation. Assume that $a(t)$ and $b(t)$ are continuous on some time interval T. Now we can transform (22.6) as follows:

Step 1 Multiply both sides of (22.6) by y^{-n}. We end up with

$$y^{-n}\dot{y} + a(t)y^{1-n} = b(t). \tag{22.7}$$

Step 2 Now define a new variable $x = y^{1-n}$. Taking the derivative of x with respect to time t, we get

$$\dot{x} = (1-n)y^{-n}\dot{y}. \tag{22.8}$$

Step 3 Using the definition in step 2, differential equation (22.7) can be written as

$$\dot{x} + (1-n)a(t)x = (1-n)b(t). \tag{22.9}$$

This is first-order **linear** nonautonomomous differential equation, which can be solved by using techniques learned in the previous chapter.

Step 4 Once we have solved for $x(t)$, we make use of definition in step 2, $x = y^{1-n}$ and derive the solution for $y(t)$.

Remark: This procedure is valid only when $y(t) \neq 0$ forall $t \in T$.

Solow Growth Model With Technical Change

Earlier, we considered Solow growth model without technical change. Now, we introduce technical change. Now suppose that production depends on capital, k, and **effective labor**, EL defined as

$$EL(t) = E(t)L(t)$$

where $E(t)$ is a measure of technology. Suppose $E(t)$ evolves as follows

$$\dot{E} = \lambda E(t), \ \lambda > 0.$$

Such technical change is called **labor augmenting**. Production function is

$$Y(t) = F(K(t), EL(t)) = K(t)^{\alpha} EL(t)^{1-\alpha}.$$

Rest of the model is identical to the previous one. Now define capital-effective labor ratio as, $\kappa = \frac{K}{EL}$. Then, given constant returns to scale, output per effective labor unit, $y(t)$ can be written as

$$y(t) = \frac{Y(t)}{EL(t)} = F(\frac{K(t)}{EL(t)}, 1) = \kappa(t)^{\alpha}.$$

Also

$$\frac{\dot{K}}{EL(t)} = \frac{sY(t) - \delta K(t)}{EL(t)} = sy(t) - \delta\kappa(t).$$

$$\dot{\kappa} = \frac{\dot{K}}{EL(t)} - \kappa\frac{\dot{E}L}{EL(t)}$$

Combining the above two equations, we get differential equation in capital-effective labor ratio which characterizes Solow growth model with technical change.

$$\dot{\kappa} + (\delta + \lambda + n)\kappa(t) = s\kappa(t)^{\alpha}.$$

As we can see that it is a Bernoulli equation. In order to solve this multiply both sides by $\kappa^{-\alpha}$ and define $x = \kappa^{1-\alpha}$. Then the above equation can be transformed into

$$\dot{x} + (\delta + \lambda + n)(1 - \alpha)x(t) = (1 - \alpha)s.$$

The solution is

$$x(t) = \frac{s}{\delta + \lambda + n} + C \exp^{-(\delta+\lambda+n)(1-\alpha)t}.$$

In terms of $\kappa(t)$, we get

$$\kappa(t) =$$

$$\left[\frac{s}{\delta + \lambda + n} + C \exp^{-(\delta+\lambda+n)(1-\alpha)t}\right]^{\frac{1}{1-\alpha}}.$$

Steady-state $\kappa(t)$ is given by

$$\overline{\kappa} = \left[\frac{s}{\delta + \lambda + n}\right]^{\frac{1}{1-\alpha}}.$$

Implications

1. Ultimately economy reaches steady-state just as in case of no technical progress.

2. At the steady state output, Y, consumption, C, and capital stock, K, grow at the rate of $\lambda + n$.

3. At the steady state per-worker output, $\frac{Y}{L}$, consumption per worker, $\frac{C}{L}$, and capital stock per worker, $\frac{K}{L}$, grow at the rate of λ.

Separable Equations

The nonautonomous equation is

$$\dot{y} = f(t, y). \tag{22.10}$$

$f(t, y)$ can always be written as the ratio of two other functions, $M(t, y)$, and $-N(t, y)$. We can then rewrite (22.10) as

$$M(t, y) + N(t, y)\dot{y} = 0. \qquad (22.11$$

> **Definition :** A non-linear, first-order differential equation is **separable** if $M(t, y) = A(t)$, a function of only t, and $N(t, y) = y$, a function of only y. A separable, nonlinear, first-order differential equation can therefore be written as

$$A(t) + b(y)\dot{y} = 0. \tag{22.12}$$

(22.12) can be solved by direct integration. (22.12) can be written as

$$A(t)dt + b(y)dy = 0. \tag{22.13}$$

This equation can be integrated directly to obtain

$$\int A(t)dt + \int b(y)dy = C. \tag{22.14}$$