1. Introduction

In the previous lectures either we directly specified behavioral relations (e.g. IS-LM model) or just considered static environment (e.g. Lucas imperfect information model or Blanchard and Kiyotaki model) or both (IS-LM model). Neither of them are very satisfactory approach to study monetary issues for various reasons. The first approach raises questions such as (i) are these behavioral relations consistent with optimizing behavior of households and firms, (ii) would these behavioral relations change if policies change (Lucas critique)? The problem with the second approach is that money is an asset. When we make decision to hold money, we have to necessarily take into account its future return as well as returns of competing assets.

In order to address these questions, we need dynamic general equilibrium (DGE) models. Now monetary issues (for that matter macroeconomic issues) are primarily analyzed using DGE models. The aim of this lecture is to introduce you to dynamic general equilibrium monetary models, which are somewhat special.

In a frictionless economies (for example where we do not have problem of double coincidence of wants) there is no role for money as a medium of exchange. Money can potentially be a medium of saving (an asset or store of value). However, agents in such economy have access to other types of assets (e.g. bonds). In this case, agents would willingly hold money only when the rate of return on money is at least as much as the return on other assets. What is rate of return on money? It is simply \( \frac{p_t}{p_{t+1}} \) where \( p_t \) is the price level at time \( t \). We will see that the rate of return on money cannot be as high as the rate of return on riskless bond. In frictionless economies, agents would not hold money unless \( p_t = \infty \). In other words the purchasing power of money \( \frac{1}{p_t} = 0 \) or money would have no value.

In order to have valued money, one needs to modify the environment. There are many ways of doing this. We will consider two approaches, which are widely used to
examine monetary issues. Both approaches assume that exchanges are not costless (they may require time, effort etc due to the problem of double coincidence of wants.) and money can reduce the costs involved in exchanges. Also money is superior medium of exchange in the sense that it is the most liquid asset (converting money in other assets and vice versa is relatively easy). Thus money is a unique asset which provides liquidity services.

In order to model the medium of exchange function of money, the first approach assumes that money yields utility and incorporates money directly into the utility function of agents. Such models are known as **money in utility function models**. The second approach assumes that certain goods and services can be bought only by using money. The constraint that goods can be bought only by using money is called **cash-in-advance** or **Clower’s constraint**. Such models are known as **cash-in-advance models**.

## 2. Money in The Utility Function Model

The motivation for putting money in the utility function is that the use of money reduces the time and effort spent on buying and selling goods and services. Higher the amount of real money balances, easier is to do transactions and the time and effort saved by agents would be more. Thus it is usually assumed that utility is strictly increasing function of real money balances.

### Example 1

Let us suppose that the representative agent has following preference

\[
\sum_{t=0}^{\infty} \beta^t \left[ \ln c_t + \gamma \ln \frac{m_{t+1}}{p_t} \right].
\]  

(2.1)

where \(m_{t+1}\) is the demand for nominal money balance at time \(t\). The representative agent receives an endowment of \(y_t\) units of non-storable good at the beginning of each period \(t\). Let \(0 < y_t < \infty\) \(\forall t\). In addition, at time 0 the representative agent is endowed with \(M_0\) units of money and at the beginning of each subsequent period, he receives \(\tau_t\) units of money as a lump-sum transfer from the government. The supply of money per agent at time 0 is then \(M_0\) and the supply of money per agent at any \(t > 0\), \(M_t = M_{t-1} + \tau_t\).
Suppose that there is an asset market where agents can buy and sell two types of financial assets— one period nominal bonds and one period real bonds (indexed to next period price). A nominal bond pays one unit of money next period, while a real bond pays one unit of good. Now we want to characterize allocations and prices in such an economy.

Let \( B_{t+1} \) be the units of nominal bond demanded by the agent at time \( t \) and \( Q_t \) be the price of the bond in nominal terms. Similarly, let \( H_{t+1} \) be the units of real bond demanded by the agent at time \( t \) and \( p_t q_t \) be the price of the bond in nominal terms. Then time \( t \) budget constraint of the representative agent can be written as

\[
c_t + \frac{m_{t+1}}{p_t} + q_t H_{t+1} + \frac{Q_t B_{t+1}}{p_t} \leq \frac{m_t}{p_t} + y_t + \frac{B_t}{p_t} + \tau_t + H_t = W_t, \quad \forall \ t
\]

(2.2)
given \( m_0 \equiv M_0 > 0 \) and \( B_0, \ H_0, \ & \tau_0 = 0 \). The representative agents’s problem is to choose the sequence of \( \{c_t, \ m_{t+1}, \ H_{t+1}, \ B_{t+1}\} \) in order to maximize his inter-temporal utility (2.1) subject to his budget constraint (2.2) given prices \( \{p_t, \ Q_t, \ q_t\} \). We can state this problem as

\[
\max_{c_t, m_{t+1}, H_{t+1}, B_{t+1}} \sum_{t=0}^{\infty} \beta^t \left[ \ln c_t + \gamma \ln \frac{m_{t+1}}{p_t} + \lambda_t \left\{ W_t - c_t - \frac{m_{t+1}}{p_t} - \frac{Q_t B_{t+1}}{p_t} - q_t H_{t+1} \right\} \right].
\]

(2.3)

The first order conditions are:

\[
c_t : \quad \frac{1}{c_t} = \lambda_t \quad (2.4)
\]

\[
m_{t+1} : \quad \frac{\gamma}{m_{t+1}} = \frac{\lambda_t}{p_t} - \beta \frac{\lambda_{t+1}}{p_{t+1}}; \quad (2.5)
\]

\[
B_{t+1} : \quad \frac{Q_t \lambda_t}{p_t} = \beta \frac{\lambda_{t+1}}{p_{t+1}}; \quad (2.6)
\]

\[
H_{t+1} : \quad q_t \lambda_t = \beta \lambda_{t+1}. \quad (2.7)
\]

From (2.4) and (2.6), we have
\[ Q_t = \beta \frac{c_t p_t}{c_{t+1} p_{t+1}} \equiv \frac{1}{I_t} \]  

(2.8)

where \(I_t\) is the gross nominal rate of interest. (2.8) shows that the price of the nominal bond, \(Q_t\), equates the marginal cost and the marginal benefit of buying a nominal bond. In general, an asset price is such that the marginal cost and the benefit of buying the asset is equal.

(2.4) and (2.7) show that the price (in real terms) of a risk-free real bond which pays one unit of good next period in real terms, \(q_t\), satisfies

\[ q_t = \beta \frac{c_t}{c_{t+1}} \equiv \frac{1}{R_t} \]  

(2.9)

where \(R_t\) is the gross real rate of interest. If we define the gross inflation rate as \(\Pi_t = \frac{p_{t+1}}{p_t}\), then using (2.8) and (2.9) we get

\[ I_t = R_t \Pi_t \]  

(2.10)

which is the **Fisher relation** in the deterministic environment. (2.4) and (2.5) imply that

\[ \frac{\gamma}{m_{t+1}} + \frac{\beta}{p_{t+1}c_{t+1}} = \frac{1}{p_t c_t}. \]  

(2.11)

With some manipulation we can show that the demand for real money balance satisfies

\[ \frac{m_{t+1}}{p_t} = \gamma c_t \left[ \frac{1}{1 - Q_t} \right]. \]  

(2.12)

(2.12) shows that the demand for real money balances is strictly positive and well defined only when \(Q_t < 1\). Since \(Q_t = \frac{1}{I_t}\), this implies that the nominal rate of interest \(I_t > 1\). (2.12) can be alternatively written as

\[ \frac{m_{t+1}}{p_t} = \gamma c_t \left[ \frac{1}{1 - \frac{p_t}{p_{t+1}} R_t} \right]. \]  

(2.13)

From (2.13) one can immediately see that in order to have well defined demand for money balances one requires \(\frac{p_t}{p_{t+1}} < R_t\). In other words, the return on money must be less than
the return on real bond \((why?)\). In this economy, money is dominated in the rate of return by the real bond.

So far we have been analyzing the optimal behavior of the representative agent. Let us now characterize the equilibrium allocations and prices. The goods market clearing condition is

\[ c_t = y_t, \forall \ t \]  \hspace{1cm} (2.14)

which pins down the optimal consumption. Initially suppose that \(\tau_t = 0\) for all \(t\) (no growth in money supply). Then the asset market clearing requires

\[ H_{t+1} = 0, \ B_{t+1} = 0, \ & \ m_{t+1} = M_0, \ \forall \ t \]  \hspace{1cm} (2.15)

which gives us equilibrium path of nominal bond and money holdings. From (2.9) and (2.15) it is immediately clear that neither consumption nor the price or rate of return on real bond is affected by changes in the initial money supply \(M_0\). In other words, the real variables are independent of initial money supply. This property of the model is known as neutrality of money.

In order to find out the path of prices, \(p_t\), we use (2.11). Define \(z_t = \frac{1}{p_t c_t}\), then (2.11) can be written as

\[ z_t = \beta z_{t+1} + \frac{\gamma}{M_0}. \]  \hspace{1cm} (2.16)

Solving (2.16) one period forward, we get

\[ z_t = \frac{\gamma}{M_0} + \beta \frac{\gamma}{M_0} + \beta^2 z_{t+2}. \]  \hspace{1cm} (2.17)

Through forward iteration we have

\[ z_t = \frac{\gamma}{M_0} [1 + \beta + \beta^2 + \ldots..] \]  \hspace{1cm} (2.18)

assuming that \(\lim_{t \to \infty} \beta^t z_t = 0\). From (2.18) we have
\[ z_t = \frac{\gamma}{M_0} \frac{1}{1 - \beta} \]  
(2.19)

which gives

\[ p_t = \frac{1 - \beta M_0}{\gamma y_t}, \forall, t. \]  
(2.20)

From (2.20) it is immediately clear that for any change in initial money supply, \( M_0 \), the price level \( p_t \) changes in the same proportion for all \( t \). The inflation rate is given by

\[ \frac{p_{t+1}}{p_t} \equiv \Pi_t = \frac{y_t}{y_{t+1}}. \]  
(2.21)

Using (2.8) and (2.21) we can find out the equilibrium path of the price of nominal bond \( Q_t \).

**The Effect of Monetary Growth**

Now let \( \tau_t = (g - 1)M_{t-1} \), where \( g \) is the growth rate of money supply per agent. Then money supply per agent in any period \( t \) is given by \( M_t = M_{t-1} + \tau_t = gM_{t-1} = g^t M_0 \). The equality of demand and supply of money requires \( m_{t+1} = M_t, \forall t \). The introduction of growth in money supply does not affect any of the real variables. All the real variables are independent of changes in the growth rate of money supply. This property of the model is known as **superneutrality of money**. Changes in the growth rate of money affect only the nominal variables. Now

\[ z_t = \frac{\gamma}{g^t M_0} + \beta z_{t+1}. \]  
(2.22)

Solving \( z_t \) one period forward we get

\[ z_t = \frac{\gamma}{g^t M_0} + \beta \frac{\gamma}{g^{t+1} M_0} + \beta^2 z_{t+2}. \]  
(2.23)

Solving (2.23) forward repeatedly we have

\[ z_t = \frac{\gamma}{g^t M_0} \left[ 1 + \frac{\beta}{g} + \left( \frac{\beta}{g} \right)^2 + \ldots \right]. \]  
(2.24)
From (2.24) we have

\[ z_t = \frac{\gamma}{M_0 g^t} \left[ \frac{g}{g - \beta} \right] \]  

which implies

\[ p_t = \frac{M_0 g^{t-1} (g - \beta)}{\gamma y_t}. \]  

The inflation rate in this case is given by

\[ \frac{p_{t+1}}{p_t} = \frac{g y_t}{y_{t+1}}. \]

Exercise: Suppose the period utility function is non-separable in consumption and real money balances \( u(c_t, \frac{m_{t+1}}{p_t}) \). Show that the price and thus return on on period real bond is affected by the monetary policy.

3. The Optimum Quantity of Money

Example 2

In order to find out the optimum quantity of money, we set up the social planner problem. The social planner maximizes

\[
\max_{c_t, m_{t+1}} \sum_{t=0}^{\infty} \left( \ln c_t + \gamma \ln \frac{m_{t+1}}{p_t} \right)
\]

subject to
\[ c_t \leq y_t, \quad \forall \ t \]  

Letting \( \lambda_t \) be the Langrangian multiplier on the resource constraint, the first order conditions for this problem are

\[ \frac{1}{c_t} = \lambda_t, \quad \forall \ t \]  

\[ \frac{\gamma}{m_{t+1}} = 0, \quad \forall \ t. \]  

(3.4) implies that at the socially optimum level of real money balances, the marginal utility of money is zero. Using the resource constraint we have

\[ c_t = y_t, \quad \forall \ t. \]  

(3.4) implies that the socially optimal real money balances is

\[ \frac{m_{t+1}}{p_t} = \infty. \]  

Comparing (3.6) with the market demand for money given by (2.12), we see that the socially optimal level of real money balances coincides with the market demand iff the price of nominal bond \( Q_t = 1 \). Since \( Q_t = \frac{1}{P_t} \), this implies that the nominal rate of interest be 1 or the net nominal rate of interest be zero. This result is called the **Friedman rule**. Using the Fisher relation this also implies that \( R_t * \Pi_t = 1 \). In other words, the return on money, \( \frac{p_t}{p_{t+1}} \), should be equal to the the return on the real bond, \( R_t \). In the stationary state when \( y_t = y \) for all \( t \), this condition reduces to \( \Pi_t = \beta \). Thus the policy maker should deflate at the rate equal to the rate of discount. Since in the stationary state, \( \Pi_t \) is equal to the rate of growth of money supply \( g \), the Friedman rule is also stated as the rate of money growth being equal to the rate of discount.

The logic behind the Friedman rule is quite simple. At the socially optimal level of real money balances, the social marginal benefit is equal to the social marginal cost of creating money. Since the creation of money is costless, the social marginal benefit should
also be zero. This is achieved when the net rate of interest is zero. An alternative way of thinking about the Friedman rule is that the opportunity cost of holding money is the net nominal rate of interest. Since money can be produced costlessly, it should be made available to agents without cost.

The Welfare Cost of Inflation

The welfare cost of inflation can be measured as the area under money demand curve. The demand for money function is given by

\[ x_t \equiv \frac{m_{t+1}}{p_t} = \gamma c_t \left( 1 + \frac{1}{i_t} \right) \] (3.7)

where \( i_t = I_t - 1 \). In the stationary environment (3.7) can be rewritten as

\[ x = \gamma y \left( 1 + \frac{1}{i} \right). \] (3.8)

Suppose that due to high inflation the net nominal rate of interest rises from \( i_1 \) to \( i_2 \) and the demand for real money balance falls from \( x_1 \) to \( x_2 \). Then the welfare loss is given by

\[ \gamma y \int_{i_1}^{i_2} \left( 1 + \frac{1}{i} \right) di = \gamma y \left[ i_2 - i_1 + \ln(i_2/i_1) \right] \] (3.9)

4. Capital Accumulation and Superneutrality

Example 3

In the previous example, we considered endowment economy. In this section, we extend our analysis to production economy. Let us suppose that the representative agent owns a firm with production technology \( f(k_t) \) where \( k_t \) is the capital stock at time \( t \). Assume that the depreciation rate \( \delta = 1 \), \( f_1(k_t) > 0 \) and \( f_{11}(k_t) < 0 \). The household optimization problem is

\[ \max_{c_t, m_{t+1}, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u \left( c_t, \frac{m_{t+1}}{p_t} \right) \] (4.1)
subject to
\[ c_t + k_{t+1} + \frac{m_{t+1}}{p_t} \leq f(k_t) + \frac{m_t}{p_t} + \frac{\tau_t}{p_t} = W_t \tag{4.2} \]
given \( m_0 > 0, k_0 > 0, \) and \( \tau_t = (g - 1)M_{t-1} \forall t \geq 1. \) I have not introduced financial assets. But one can easily do so and their prices will be given by Euler equations as in example 1 of this lecture. This optimization problem can be recast as

\[ \max_{c_t, m_{t+1}, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \left[ u \left( c_t, \frac{m_{t+1}}{p_t} \right) + \lambda_t \left( W_t - c_t - k_{t+1} - \frac{m_{t+1}}{p_t} \right) \right] \tag{4.3} \]

The first order conditions are

\[ c_t : u_1 \left( c_t, \frac{m_{t+1}}{p_t} \right) = \lambda_t \tag{4.4} \]

where \( u_1() \) is the marginal utility of consumption.

\[ k_{t+1} : \beta f_1(k_{t+1}) \lambda_{t+1} = \lambda_t \tag{4.5} \]

\[ m_{t+1} : \frac{u_2 \left( c_t, \frac{m_{t+1}}{p_t} \right)}{p_t} + \beta \frac{\lambda_{t+1}}{p_{t+1}} = \frac{\lambda_t}{p_t} \tag{4.6} \]

where \( u_2() \) is the marginal utility from real money balance. (4.4) and (4.5) imply

\[ u_1 \left( c_t, \frac{m_{t+1}}{p_t} \right) = \beta u_1 \left( c_{t+1}, \frac{m_{t+2}}{p_{t+1}} \right) f_1(k_{t+1}). \tag{4.7} \]

(4.4) and (4.6) imply

\[ \frac{u_2 \left( c_t, \frac{m_{t+1}}{p_t} \right)}{u_1 \left( c_t, \frac{m_{t+1}}{p_t} \right)} + \beta \frac{p_t}{p_{t+1}} \frac{u_1 \left( c_{t+1}, \frac{m_{t+2}}{p_{t+1}} \right)}{u_1 \left( c_{t+1}, \frac{m_{t+2}}{p_{t+1}} \right)} = 1. \tag{4.8} \]

In order to find out equilibrium path of \( \{c_t, m_{t+1}, k_{t+1}, p_t\} \) we need market clearing constraints. The goods market clearing constraint is

\[ c_t + k_{t+1} = f(k_t), \forall t. \tag{4.9} \]
The money market clearing constraint is

\[ m_{t+1} = m_t + \tau_t = M_t. \]  
(4.10)

Using the definition of \( \tau_t \) we have

\[ m_{t+1} = gM_{t-1}. \]  
(4.11)

Using (4.7) to (4.11), we can solve for the equilibrium path of \( \{c_t, k_{t+1}, p_t\} \). In order to solve for these values, we will have to solve a system of difference equations.

**Stationary State**

In the stationary state, consumption level, capital stock, and real money balance are constant over time. Let \( \frac{m_{t+1}}{p_t} = x_t = x \), \( \forall \ t \). (4.7) implies that

\[ u_1(c, x) = \beta f_1(k)u_1(c, x) \]  
(4.12)

or

\[ f_1(k) = \beta^{-1}. \]  
(4.13)

(4.13) implies that the stationary state capital stock is completely determined by the discount rate and is independent of monetary policy. If \( k \) is independent of monetary policy this implies that consumption \( c \) is also independent of monetary policy. One can show that the price of real bond satisfies

\[ q = \beta \]  
(4.14)

which is also independent of monetary policy. This is the case though the utility function is non-separable (in contrast to the previous exercise). This happens because in this economy agents can save in terms of capital stock and thus real bond must pay as much return as the capital stock. In the stationary state, of course, the rate of inflation \( \frac{p_{t+1}}{p_t} \) is equal to the rate of growth of money supply \( g \).
Short Run Effects

In the short run, changes in monetary policy can affect real variables. Defining the rate of return on money as \( R_{mt} = \frac{p_t}{p_{t+1}} \) and combining (4.7) and (4.8) we have

\[
\frac{u_2 \left( c_t, \frac{m_{t+1}}{p_t} \right)}{u_1 \left( c_t, \frac{m_{t+1}}{p_t} \right)} + \frac{R_{mt}}{f_1(k_{t+1})} = 1
\]  

(4.15)

(4.15) can be written as

\[
f_1(k_{t+1}) = \frac{R_{mt}}{1 - MRS} = R_{mt} \left[ 1 - \frac{u_2 \left( c_t, \frac{m_{t+1}}{p_t} \right)}{u_1 \left( c_t, \frac{m_{t+1}}{p_t} \right)} \right]^{-1}
\]  

(4.16)

From (4.16) one can immediately see that the monetary policy can affect capital accumulation by affecting \( R_{mt} \) and \( \frac{u_2 \left( c_t, \frac{m_{t+1}}{p_t} \right)}{u_1 \left( c_t, \frac{m_{t+1}}{p_t} \right)} \).

Holding \( \frac{u_2 \left( c_t, \frac{m_{t+1}}{p_t} \right)}{u_1 \left( c_t, \frac{m_{t+1}}{p_t} \right)} \) or the marginal rate of substitution between real money balance and consumption constant, an increase in money supply increases the inflation rate and thus reduces the return on money \( R_{mt} \). This implies that \( f_1(k_{t+1}) \) must fall which is possible only when \( k_{t+1} \) rises. Thus expansionary monetary policy in the short run can increase capital accumulation. This is called the Tobin effect. Idea is that higher inflation reduces the return on money inducing agents to shift away from real money balances to capital leading to higher capital accumulation and output.

Now suppose \( R_{mt} \) is constant but not the marginal rate of substitution (MRS). Also suppose that the MRS is a decreasing function of real money balance i.e. lower real money balances increases MRS. In this case, expansionary monetary policy by reducing real money balances and increasing the MRS increases the RHS of (4.16). This implies that the LHS must rise which can happen only when \( k_{t+1} \) falls. Thus expansionary monetary policy has negative impact on capital accumulation and output in the short run. This second effect is called the real money balance effect. The idea is that expansionary monetary policy through inflation reduces the wealth of agents leading to a fall in the aggregate demand. The fall in the aggregate demand reduces capital accumulation and output in the short run.
To summarize, the effect of monetary policy in the short run depends on

(1.) Is the MRS between consumption and real money balance decreasing function of real money balance?

(2.) Does the Tobin effect dominate the real money balance effect?

5. Cash-In-Advance Models

CIA models emphasize the role of money as a medium of exchange. They directly posit that buying and selling of certain goods and services require use of money. In order to acquire these goods and services one has to acquire money in advance. This constraint on transactions is known as cash-in-advance constraint. The subset of goods and services on which this constraint applies varies from model to model. In some models, purchases of only consumption goods are subject to CIA constraint, in some models purchases of investment goods can also be subject to CIA constraint and so on. Unfortunately, the results of the models normally depend on the exact specification of the CIA constraint. The goods whose purchase requires acquisition of money in advance (or goods subject to CIA constraint) are known as cash goods. The goods whose purchase are not subject to CIA constraint is known as credit goods.

In such models, in any time period temporary separation is made between the goods and the financial (or asset) market. Agents cannot participate in both the markets simultaneously. In any time period, either they first trade in financial (or asset) market then in the goods market or vice-versa. Let us now consider a representative agent CIA economy with production.

Example 4

Suppose that consumption is cash good and capital is credit good. So CIA constraint applies only on consumption good. The preference of the representative agent is given by

\[ U = \sum_{t=0}^{\infty} \beta^t \ln c_t. \]  

(5.1)
The representative agent also owns a firm with production technology \( f(k_t) \) where \( k_t \) is the capital stock available at the beginning of period \( t \). Let \( f'(\cdot) > 0 \) and \( f''(\cdot) < 0 \). Suppose that the depreciation rate \( \delta = 1 \).

Assume that the agent enters period \( t \) with the nominal balances equal to \( m_t \) that are carried from the previous period. In addition, these balances are augmented with a lump-sum-transfer equal to \( \tau_t \) at the beginning of period \( t \). The agent during any period is required to use these **previously acquired** nominal balances to buy consumption good. The purchase of consumption good at time \( t \) is subject to the CIA constraint

\[
p_t c_t \leq m_t + \tau_t \quad (5.2)
\]

where \( p_t \) is the price of consumption good. By assumption the agent cannot use current nominal receipts to purchase consumption good in the current period.

Separate from consumption goods market, there is an asset market where the agents can buy and sell one period nominal riskless bond and capital stock. Purchases in the asset market is not subject to CIA constraint. The agents can use current receipts to finance purchases in the asset market. Also suppose that transactions in the asset market takes place after transactions have been completed in the consumption goods market. The budget constraint for the asset market in any period \( t \) is

\[
m_{t+1} + p_t k_{t+1} + Q_t B_{t+1} \leq (m_t + \tau_t - p_t c_t) + B_t + p_t f(k_t) \quad (5.3)
\]

where \( B_{t+1} \) is the total units of nominal bond demanded at time \( t \) and \( Q_t \) is the nominal price of the bond. The first term in the RHS is the nominal money balances unspent in the consumption goods market. Second and third terms are sales receipts and total return from previous bond holding. (5.3) can be written as

\[
c_t + k_{t+1} + \frac{m_{t+1}}{p_t} + \frac{Q_t B_{t+1}}{p_t} \leq \frac{m_t + \tau_t}{p_t} + f(k_t) + \frac{B_t}{p_t} \equiv W_t. \quad (5.4)
\]

The representative agents’s optimization problem is to choose the sequence of \( \{c_t, k_{t+1}, m_{t+1}, B_{t+1}\} \) in order to maximize his inter-temporal utility (5.1) subject to the CIA constraint (5.2), and the asset market budget constraint (5.4) taking as given prices.
\{p_t, Q_t\}$. The agent’s maximization problem can be recast as

$$\max_{c_t, m_{t+1}, k_{t+1}, B_{t+1}} \sum_{t=0}^{\infty} \beta^t \ln c_t + \sum_{t=0}^{\infty} \beta^t \left[ \lambda_{ct} \left( \frac{m_t + \tau_t}{p_t} - c_t \right) + \lambda_{mt} \left( W_t - c_t - k_{t+1} - \frac{m_{t+1}}{p_t} - \frac{Q_t B_{t+1}}{p_t} \right) \right]. \quad (5.5)$$

The first order conditions are

$$c_t : \frac{1}{c_t} = \lambda_{ct} + \lambda_{mt} \quad (5.6)$$

(5.6) is different from (2.4) (foc in the case of MIU model) in the sense that the marginal utility of consumption exceeds the marginal utility of real income or money. This happens because of cash-in-advance constraint. One additional unit of money not only relaxes constraint on real money balances, but it also relaxes CIA constraint.

$$m_{t+1} : \frac{\lambda_{mt}}{p_t} = \beta \frac{p_t}{p_{t+1} c_{t+1}} \left[ \lambda_{ct+1} + \lambda_{mt+1} \right] \quad (5.7)$$

$$k_{t+1} : \beta \lambda_{mt+1} f_1(k_{t+1}) = \lambda_{mt} \quad (5.8)$$

$$B_{t+1} : \beta \frac{\lambda_{mt+1}}{p_t} = Q_t \frac{\lambda_{mt}}{p_t} \quad (5.9)$$

Combining (5.6) and (5.7) we have

$$\lambda_{mt} = \beta \frac{p_t}{p_{t+1} c_{t+1}}. \quad (5.10)$$

Combining (5.10) and (5.8) we have

$$\frac{p_t}{c_{t+1} p_{t+1}} = \beta \frac{p_{t+1}}{c_{t+2} p_{t+2}} f'(k_{t+1}) \quad (5.11)$$

(5.6) and (5.9) give us
Combining (5.12) and (5.7), we have

\[ Q_t = \beta \frac{p_t}{p_{t+1}} \frac{\lambda_{mt+1}}{\lambda_{mt}}. \]  

(5.12)

Let \( i_t \) and \( I_t \) be the net and gross nominal interest rates respectively. Then from (5.13) we get

\[ i_t = \frac{\lambda_{ct+1}}{\lambda_{mt+1}} \quad \forall \ t. \]  

(5.14)

(5.14) suggests that CIA constraint is binding \((\lambda_{ct} > 0)\) if and only if the net nominal rate of interest \( i_t > 0 \). This condition can be alternatively stated as follows. From Fisher relation (which can be easily shown to hold in this economy) we know that

\[ I_t = \frac{p_{t+1}}{p_t} R_t \]  

(5.15)

where \( R_t \) is the gross return on real bond. Then CIA constraint binds if and only if \( \frac{p_{t+1}}{p_t} R_t > 1 \). This implies that

\[ R_t > \frac{p_t}{p_{t+1}}. \]  

(5.16)

In other words in this economy as well money is dominated in the rate of return. If CIA constraint is binding, money provides liquidity service. Binding CIA constraint also implies that the consumption velocity of money defined as \( \frac{p_t c_t}{M_{t-1}} \) is equal to one. One can also show that money is valued \((\lambda_{mt} > 0)\) if and only if CIA constraint is binding.

Let \( \tau_t = (g - 1)M_{t-1} \) where \( M_{t-1} \) is the per agent money supply of money in period \( t - 1 \). The total supply of money per agent at time \( t \) is \( M_t = gM_{t-1} \). Then using (5.11) together with the market clearing constraints

\[ c_t + k_{t+1} = f(k_t), \quad m_{t+1} = M_t = gM_{t-1}, \quad \& \quad B_t = 0 \]  

(5.17)

and the cash-in-advance constraint (5.2), one can derive the equilibrium path of
\{c_t, \ k_{t+1}, \ m_{t+1}, \ B_{t+1}, \ p_t, \ Q_t\}. \ Using \ cash-in-advance \ constraint \ and \ (5.17) \ we \ can \ rewrite \ (5.11) \ as
\[
\frac{g}{\beta} = \frac{p_{t+1}}{p_t} f_1(k_{t+1})
\] (5.18)

Further CIA constraint and (5.17) imply that
\[
p_t = \frac{gM_{t-1}}{f(k_t) - k_{t+1}}.
\] (5.19)

Using (5.19), (5.18) can be written as
\[
\frac{1}{\beta} = \frac{f(k_t) - k_{t+1}}{f(k_{t+1}) - k_{t+2}} * f_1(k_{t+1})
\] (5.20)

which gives the path of capital stock. You can immediately see that capital stock and thus consumption and output are independent of monetary policy.

**Stationary State Analysis**

In the stationary state, consumption, output, and real money balances are constant over time. In this case, (5.20) reduces to
\[
f_1(k) = \beta^{-1}.
\] (5.21)

(5.21) implies that money is superneutral. However, the result that real variables are independent of monetary policy depends on the exact specification on utility function and as well as the goods for which cash is required.

**Exercise:** In the above example, assume that the agents require cash to buy both consumption and investment goods i.e. \(c_t + k_{t+1} \leq \frac{m_{t+1} + r_t}{p_t}\). Show that money is no longer superneutral.
Example 5

A Cash-In-Advance Model with Leisure

Now we modify the environment of the previous example by introducing labor-leisure choice. Suppose that each representative agent is endowed with 1 unit of labor every period. Suppose that the sub-period utility function is

$$\ln c_t + \gamma \ln(1 - n_t)$$ (5.22)

where \(n_t\) is labor supplied at time \(t\). The production function is given by \(f(k_t, n_t)\), which is strictly increasing and concave in both of its arguments and satisfies constant returns to scale. Also suppose that labor and capital are complimentary \(f_{12} > 0\) and the depreciation rate \(\delta = 1\). For simplicity assume that money is the only financial asset (you can introduce other assets without affecting the results). The representative agent’s optimization problem is now

$$\max_{c_t, n_t, m_{t+1}, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \left[ \ln c_t + \gamma \ln(1 - n_t) \right] +$$

$$\sum_{t=0}^{\infty} \beta^t \left[ \lambda_{ct} \left\{ \frac{m_t + \tau_t}{p_t} - c_t \right\} + \lambda_{mt} \left\{ W_t - c_t - k_{t+1} - \frac{m_{t+1}}{p_t} \right\} \right]$$ (5.23)

where

$$W_t = f(k_t) + \frac{m_{t+1} + \tau_t}{p_t}.$$ (5.24)

The first order conditions for \(c_t, m_{t+1}, k_{t+1}\) are same as in the previous example. The optimal choice for labor supply \(n_t\) is given by

$$n_t : \frac{\gamma}{1 - n_t} = \lambda_m f_2(k_t, n_t).$$ (5.25)

Putting (5.10) in (5.25) we get

$$\frac{\gamma}{1 - n_t} = \beta \frac{p_t}{p_{t+1}} \frac{f_2(k_t, n_t)}{c_{t+1}}.$$ (5.26)
In the stationary state, (5.26) reduces to

\[ \frac{\gamma g}{1 - n} = \frac{\beta f_2(k, n)}{f(k, n) - k}. \]  

(5.27)

We also have

\[ f_1(k, n) = \beta^{-1} \]  

(5.28)

From (5.28) you can immediately see that an increase in \( g \) reduces the labor supplied \( n \) and given the complementarity between labor and capital in the production function this also implies that capital stock falls. Thus in this case money is no longer superneutral.

The logic of this result is quite simple. In this model, consumption good is cash good and leisure is credit good. Higher inflation is a tax on cash good and thus the agents try to move away from cash good towards credit good. The result is that we have lower consumption and higher leisure (less labor supply).

Exercise: Show that the Friedman rule is optimal in the above example.